INVESTING FOR RETIREMENT THROUGH A WITH-PROFITS PENSION SCHEME: A CLIENT'S PERSPECTIVE

MICHAEL PREISEL, SØREN F. JARNER, AND RUNE ELIASEN

ATP - Danish Labour Market Supplementary Pension Scheme

Kongens Vænge 8, DK-3400 Hillerød, Denmark,

mip@atp.dk, sj@atp.dk, re@atp.dk

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Abstract. Saving for retirement by with-profits pension contracts is markedly different from traditional savings vehicles in at least two aspects: premia committed to the company are managed according to the preferences of the company – which may not coincide with the long-term interests of the client – and the return on investments is not directly transferred to client's savings but awaits a decision by the company to spend it as bonus.

We show that a management's general aversion to (short-term) insolvency risk results in investment strategies dynamically scaling investment risk up and down with the current funding status of the company. The resultant dynamics hugely impacts the long-term funding status of the pension company and thereby the investment outcome of with-profits contracts. We show that for a one-period optimizing company there exists a stationary regime only for moderately aggressive investment strategies, and we derive an analytical approximation to the stationary funding ratio distribution when it exists. In contrast to the one-period case, we show that the highest expected bonus level in stationarity is not achieved for the most aggressive investment strategy available. The reason being that if investments become too aggressive the company will spend a lot of time at low funding ratios where bonus cannot be attributed impairing the average bonus.

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1. Introduction

Modern investment management is about optimizing risk versus reward. The theoretical foundation was laid by Markowitz (1952) whose key result was that investors should diversify their investment portfolios to minimize variance of their investment without impairing expected return. The theory is easily generalized to take each investors personal preferences into account by the introduction of a utility function, Merton (1971). The main premise remains though - any gain or loss is attributed the investor solely!

For a large group of investors this paradigm is not applicable, namely those saving for their retirement through participating — or with-profits — programmes through a pension company. In this type of savings vehicle the pension company acts on behalf of its customers as an investment *collective* where each client has chosen to *participate* in the programme.

With-profits pension schemes are therefore distinctly different from other savings vehicles because stakeholders are different. After the contract has been signed the investment outcome is not driven by markets solely but has become dependent on investment and bonus decisions made by the company outside the client's control.

Another key issue to pension fund operation is *stationarity* since the horizon of client's participation is so long – life-long in fact. The alternative, ie. not to ensure stationary operation, will lead to unfair distribution of wealth amongst clients. Investment strategies therefore must be "long-term" by which we explicitly mean they should ensure stationarity *and* maximize expected bonus (with respect to the stationary distribution).

This is in potential conflict with managements objectives whose (career) horizon supposedly is much shorter. We will therefore assume management to pursue investment strategies avoiding underfunding at all times but to optimize return over shorter periods of time.

Our key finding is that an aggressive (short-term) investment strategy does not always lead to higher long-term expected bonus. When management pursues investment strategies protecting funding, such strategies are a tight balance between (dynamically) getting out of risky investments if funding becomes low and (dynamically) scaling up risky assets when funding is high. The continuous re-allocation in and out of risky assets will in general lead to higher realized bonus to clients the faster the allocation into risky assets are. In some situations, though, dynamical asset allocation becomes counter-productive such that the company is trapped at low funding levels in long periods . . . unable to attribute bonus. Such bonus lapses obviously will deteriorate long-term expected bonus to clients and can be controlled if companies voluntarily decide an "equity cap" which define an upper limit to investment risk regardless of the funding status.

1.1. Background.

Stochastic models for pension funds have been developed by several authors on defined benefit: Owadally and Haberman (2004); Cairns (1996), and on defined contribution: Briys and de Varenne (1997);

Briys and de Varenne (2001); Grosen and Jørgensen (2000); Grosen and Jørgensen (2002). These models are all finite horizon models, some of them explicitly allowing insolvency to occur which — in an infinite horizon setting — ultimately will lead to the demise of the company.

Of particular interest is the application of stochastic control to the investment decision which is a rational framework for deducing pension fund dynamics at infinite horizons. Defined benefit is treated in Cairns (2000); Steffensen (2006) for quadratic utilities which, unfortunately, lead to somewhat counterintuitive investment strategies where the company increases investment risk at low funding ratios. HARA-utilities (*Hyperbolic Absolute Risk Aversion*), as considered in this paper, are also discussed in Cairns (2000) but leads to either trivial or non-stationary fund operation.

A continuous-time framework for pension schemes taking explicitly both realized technical and financial surplus into account is derived in Norberg (1999); Steffensen (2000); Norberg (2001); Nielsen (2006). Within this framework stochastic control is applied to derive optimal bonus strategies for dividend bonus (bonus is an immediate lump sum payment to clients) as well as benefit bonus (bonus is converted to (delayed) higher guaranteed payments) for power utilities in Steffensen (2004) which is the essential reference on bonus within with-profits pension schemes.

For dividend bonus schemes it is shown that the optimal bonus strategy is to (continuously) pay a fraction of the bonus reserve as bonus where the bonus reserves are defined as the difference between the value of assets and liabilities.

The results on benefit bonus are more promising since they can be interpreted as rationalizing bonus policies as they are actually practiced in real life. Based on results on utility optimization of durable goods by Hindi and Huang (1993) it is shown, that the optimal bonus strategy is to continuously attribute bonus when the bonus reserve crosses some boundary — and then *all* funds above this barrier should be attributed.

In the model introduced below bonus will be attributed at discrete points in time if the funding ratio, ie. assets divided by liabilities, at that time is sufficiently high. This closely reflects real life decisions on bonus attribution where it is decided to spend only a part of the bonus reserve once a year. The discretization in time naturally is a compromise to the continuous case — a point supported by Brennan (1993) where it is shown that a specific class of (discrete time) bonus strategies are inefficient in optimizing terminal utility.

The sub-periods between bonus attribution also serve as the local horizon over which management optimizes return. In a finite horizon setting a terminal HARA-utility with a subsistence level corresponding to a terminal funding ratio of one is a sufficient condition for an investor to remain funded during the time-span of the whole period. Investment strategies avoiding insolvency are generally known as CPPI strategies (*Constant Proportion Portfolio Insurance*), Black and Perold (1992), and are formally treated by Cox and Huang (1989) in the unrestricted case, ie. no subsistence level, and in Teplá (2001) in the restricted case. Optimizing terminal utility in a pension saving's context is discussed in Gerber and Shiu (2000).

2. The Model

Pension products typically offered by pension companies range from standard annuities to individually tailored contracts. The primary task of the pension company is therefore to diversify insurance risks, ie. mortality/survival risk, for its clients whereas the underlying investment vehicle is common to all contracts. In this way clients of a pension fund are literally joined together by a common investment destiny — or investment collective. The investment collective must cover future guaranteed payments on individual contracts which is assured by a reserve equal to the net present value of the future cash flow. Additional surplus (the bonus reserve) also belongs to the collective but it is the company that decides when — and how much — of this to distribute as bonus. Managing the bonus reserve is therefore a main objective of the pension company.

2.1. **Capital Market.** Investment decisions must necessarily reflect market dynamics, investment opportunity, insolvency risk etc. — decisions that all relate to probabilities of certain events of the operation of the company. We will call the probability measure, $\mathbb{P}$, governing these events the *physical measure*. Later, in Sec. 3.2, we will also be concerned with *valuation* of the actions of the company and the equivalent martingale measure will be introduced.

In principle, a capital market model should encompass a wide range of assets but it is well known by the Mutual Fund Separation Theorem, Merton (1971), that for HARA-utility optimizers (that we shall be concerned with) the optimal asset allocation will always be between a risk-less asset (if it exists) and an optimal portfolio (at least for diffusions with constant parameters). For convenience we shall refer to these as 'cash' and 'equity', respectively.

We therefore formulate the model in the Black-Scholes framework by defining a financial market consisting of cash yielding a constant rate r, and equity with price process $S = (S_t)_{t\geq 0}$ following a geometric Brownian motion

$$\frac{dS_t}{S_t} = (r + \mu)dt + \sigma dW_t, \tag{1}$$

where μ is a risk premium, σ is volatility and W is standard Brownian motion under the (physical) measure, $\mathbb{P}$. Thus investing in 'risky' equity yields a higher expected return than in risk-less money.

2.2. **The Pension Company.** The company offers a range of life-insurance products. In our setting, customers insure specific lives and receive a guaranteed (minimum) payment if the insured dies before a certain date (life insurance) or the insured survives a certain date (pension).

Given assumptions of mortality rates of insured lives the future payment — or benefit — stream of the company can be computed. In general insurance, the company has to manage any risk originating from uncertainty in these cash flows but since our objective is to analyze the common savings vehicle of the insurance contracts, we will assume that realized cash flows are equal to expected cash flows. Further, we will disregard any cost, underwriting fee, or other charge and only consider the part of the premium actually entering the investment collective. In ignoring both systematic and non-systematic mortality risk we follow the approach of (Briys and de Varenne, 1997; Grosen and Jørgensen, 2000) while eg. Cairns (2000) allows for a non-systematic component.

At any point in time, $t \geq 0$, the company can compute the expected cash flow of all contracts and compute the present value. For any (expected) dollar guaranteed at (future) time T this amounts to $e^{-r(T-t)}$ at time t which is the (fair value) reserve, R_t. We will subdivide time in equally spaced periods at times $0 = t_0 < t_1 < \ldots$ when the company decides to attribute bonus (or not), cf. the discussion above. For simplicity we will assume that also premium and benefit payments fall at these times.

Under the assumptions above – realized cash flows equal expected cash flows – and a constant interest rate in the capital market model the evolution of the (prospective) reserve during a time interval can be put on *retrospective* form:

$$R_t = R_{t_i} e^{r(t-t_i)}, \quad \text{where } i = \max\{j \in \mathbb{N}_0 : t_j \leq t\}.$$

The reduction of the reserve computation to retrospective form vastly facilitates the multi-period analysis in subsequent sections.

The financial health of the investment collective is measured at any time, t, by the *funding ratio*, $F_t = A_t/R_t$, where A_t is total assets of the investment collective.

Key to the with-profits savings scheme is bonus. A scheme which has found both theoretical and practical application is to distribute a fraction of the surplus above a certain funding level as bonus, cf. (Briys and de Varenne, 1997; Grosen and Jørgensen, 2000; PGGM, 2006). As discussed in the introduction such strategies have recently been rationalized in Steffensen (2004). In fact it is found optimal continuously to distribute *all* surplus above a certain threshold as bonus.

Inspired by this we will assume — on a relative scale — the company to consider distributing bonus only at the set of times $\{t_i\}$ defined above and that the decision is based solely on the current funding ratio. If the funding ratio exceeds a certain level, κ, the surplus is distributed among the investment collective by increasing all guaranteed payments by the same percentage, r_i^B in such a way that the funding ratio *after* bonus attribution is lowered to κ; otherwise no bonus is attributed. Assuming for simplicity that any premia and benefits due exactly match we thus have

$$R_{t_i} = (1 + r_i^B) R_{t_i-}, \tag{2}$$

where R_{t_i-} and R_{t_i} is the reserve before and after bonus attribution, respectively, and

$$r_i^B = \begin{cases} 0 & F_{t_i-} \leq \kappa, \\ \frac{F_{t_i-}-\kappa}{\kappa} & F_{t_i-} > \kappa. \end{cases} \tag{3}$$

Notice that if bonus is attributed then $F_{t_i} = \kappa$, otherwise $F_{t_i} = F_{t_i-}$.

At any time, t, assets are split between equity and cash by the equity share, γ_t. The equity share is at the control of the company and — together with the market dynamics (1) — determines the dynamics of assets of the collective. We will refer to the particular choice of γ_t as the *investment strategy* of the company. The corresponding balance sheet of the company is depicted in Table 1.

Finally, the company is under supervision of some (outside) authority, or Regulator, which (continuously) monitors the funding ratio of the company. If at any time total assets equals the reserve, $A_t = R_t$, the company is declared bankrupt and policy holders immediately receive the reserve.

Assets		Liabilities	
equity	$\gamma_t A_t$	R_t	reserve
cash	$(1-\gamma_t)A_t$	$A_t - R_t$	bonus reserve
	A_t	A_t	

TABLE 1. Balance sheet of the company: Assets are split between equity and cash by the equity share, γ_t, and liabilities are split between the (mandatory) reserve, R_t, and non-distributed surplus (the bonus reserve).

Given an investment strategy, assets of the investment collective becomes a stochastic process evolving according to the stochastic differential equation

$$\frac{dA_t}{A_t} = (r + \gamma_t \mu)\, dt + \gamma_t \sigma dW_t. \tag{4}$$

Since liabilities and bonus are completely specified in terms of assets, A_t, the dynamics of the full model is completely determined by κ, γ_t and W_t (or, equivalently, S_t).

2.3. **Optimal Investment Strategy.** During operation the company continuously must decide the equity share, γ_t, by balancing risks against rewards. We will assume that the investment strategy is the result of an optimization within each sub-period, $[t_i, t_{i+1})$. To fix ideas we will work on the time interval $[0, T)$, representing a given sub-period of the multi-period model.

We will phrase the optimization as a stochastic control problem where the aim is to optimize the expected value of a utility function K. We will assume that the utility function is a function of terminal funding ratio before bonus attribution, F_{T-}, only and impose the restriction $F_t \geq 1$ for all $t \in [0, T)$, ie. management will only pursue "safe" investment strategies ensuring the survival of the company. In particular, this requires that the company starts out being funded. Investing all assets in cash is an example of a safe strategy resulting in a constant funding ratio. This strategy, however, is in general not optimal.

It turns out that requiring the funding ratio process to stay at or above 1 at all points in time is equivalent to the apparently weaker condition that only the terminal funding ratio should be greater or equal to 1. Intuitively, this is clear since once the funding ratio drops below 1 there exists no investment strategy under which the company can be funded at a later time with certainty. The optimization problem can thus be stated as finding the optimal strategy, $\hat{\gamma}$, and the corresponding funding ratio process, $F^{\hat{\gamma}}$, such that $F^{\hat{\gamma}}_{T-} \geq 1$ and

$$\mathsf{E}[K(F^{\hat{\gamma}}_{T-})] = \sup_{\gamma} \mathsf{E}[K(F^{\gamma}_{T-})], \tag{5}$$

where the supremum is taken over all strategies γ with $F^{\gamma}_{T-} \geq 1$.

For general utility functions this constrained optimization can be carried out using results of (Teplá, 2001). For utility functions with derivative satisfying $K'(1+) = \infty$, however, the optimal strategy from an *unconstrained* optimization is guaranteed to satisfy the terminal constraint. As optimal strategies from unconstrained optimizations generally have a simpler structure than their constrained counterparts we will take the latter approach since this facilitates further study of the optimal funding ratio process. Specifically, we will consider the following class of increasing, concave utility functions defined for $\nu < 1$ and $x \geq 1$

$$K_{\nu}(x) = \begin{cases} \frac{1}{\nu}(x-1)^{\nu} & \text{for } \nu \neq 0, \\ \log(x-1) & \text{for } \nu = 0, \end{cases} \tag{6}$$

where small values of ν correspond to high risk aversion and values of ν close to one correspond to low risk aversion. Using the methodology of (Cox and Huang, 1989) we then have

Theorem 2.1. *The funding ratio process solving (5) for K_{ν} is given by*

$$F^{\hat{\gamma}}_t = (f-1)e^{\frac{\tilde{\sigma}^2}{2}(1-2\nu)t + \tilde{\sigma}W_t} + 1, \tag{7}$$

where $f \geq 1$ is the initial funding ratio and $\tilde{\sigma} = \mu/(\sigma(1-\nu))$.

The optimal control is given by

$$\hat{\gamma}_t = \frac{\mu}{\sigma^2(1-\nu)} \frac{F^{\hat{\gamma}}_t - 1}{F^{\hat{\gamma}}_t}. \tag{8}$$

Proof. Problem (5) is equivalent to maximizing the expectation of $\tilde{K}_\nu(A_T^\gamma)$, where

$$\tilde{K}_\nu(a) = \begin{cases} \frac{1}{\nu}(a - R_0 e^{rT})^\nu & \text{for } \nu \neq 0, \\ \log(a - R_0 e^{rT}) & \text{for } \nu = 0. \end{cases}$$

It follows from (Cox and Huang, 1989) that the optimal asset process is given by

$$\begin{aligned} A_t^{\hat{\gamma}} &= G(Z_t, t) \\ &= Z_t\, \mathsf{E}[\tilde{K}_\nu'^{-1}(Z_T^{-1}) Z_T^{-1} | Z_t] \\ &= Z_t\, \mathsf{E}[(Z_T^{1/(1-\nu)} + R_0 e^{rT}) Z_T^{-1} | Z_t], \end{aligned}$$

where Z satisfies $dZ_t = (r + \xi^2) Z_t dt - \xi Z_t dW_t$ with $\xi = -\mu/\sigma$. Inserting the solution $Z_T = Z_t \exp((r + \xi^2/2)(T-t) - \xi(W_T - W_t))$ yields

$$\begin{aligned} G(Z_t, t) =& Z_t^{1/(1-\nu)}\, \mathsf{E}[e^{(\frac{\nu}{1-\nu})(r+\xi^2/2)(T-t) - \frac{\nu\xi}{1-\nu}(W_T - W_t)}] \\ &+ R_0 e^{rT}\, \mathsf{E}[e^{-(r+\xi^2/2)(T-t)+\xi(W_T-W_t)}] \\ =& Z_t^{1/(1-\nu)} e^{\left(\frac{\nu r}{1-\nu} + \frac{\nu\xi^2}{2(1-\nu)^2}\right)(T-t)} + R_0 e^{rt}. \end{aligned}$$

By use of the boundary condition $fR_0 = G(Z_0, 0)$ we have

$$fR_0 = Z_0^{1/(1-\nu)} e^{\left(\frac{\nu r}{1-\nu} + \frac{\nu\xi^2}{2(1-\nu)^2}\right)T} + R_0,$$

and thereby

$$\begin{aligned} G(Z_t, t) &= Z_0^{1/(1-\nu)} e^{(\frac{r}{1-\nu} + \frac{\xi^2}{2(1-\nu)})t - \frac{\xi}{1-\nu} W_t} e^{\left(\frac{\nu r}{1-\nu} + \frac{\nu\xi^2}{2(1-\nu)^2}\right)(T-t)} + R_0 e^{rt} \\ &= R_0(f-1) e^{rt + \frac{\xi^2}{2(1-\nu)^2}(1-2\nu)t - \frac{\xi}{1-\nu} W_t} + R_0 e^{rt}. \end{aligned}$$

Dividing with the reserve, $R_0 e^{rt}$, gives (7).

Further, it follows from (Cox and Huang, 1989) that the optimal control is given by

$$\hat{\gamma}_t = \frac{\mu}{\sigma^2} \frac{G_Z(Z_t, t) Z_t}{G(Z_t, t)} = \frac{\mu}{\sigma^2(1-\nu)} \frac{A_t^{\hat{\gamma}} - R_0 e^{rt}}{A_t^{\hat{\gamma}}} = \frac{\mu}{\sigma^2(1-\nu)} \frac{F_t^{\hat{\gamma}} - 1}{F_t^{\hat{\gamma}}}.$$

□

In our setup investment strategies are derived from a short-term optimization of investment return (over the sub-periods defined by bonus attribution) and therefore parametrized by the risk-aversion parameter, ν. Later, we will show that higher ν does not necessarily lead to higher expected bonus.

Therefore, it is the company's (not it's clients) risk-aversion that defines ν: The smaller ν the more risk-averse the company is. Although well-defined in theory determination of risk-aversion in real life can be complicated. Some insight into the rôle of the risk-aversion parameter can be found from the limiting equity share – or *equity cap*:

$$\hat{\gamma}(\infty) = \lim_{f\to\infty} \hat{\gamma}(f) = \frac{\mu}{\sigma^2(1-\nu)}. \tag{9}$$

The risk-aversion parameter basically determines the maximum equity share the company is allowed to hold, that is, it effectively determines an overall cap to the (dynamic) allocation to equity. In real life most asset managers would have a clear attitude to the maximal acceptable equity share thereby indirectly quantifying their aversion to risk.

The equity cap is equal to the constant allocation to equity the company would hold if acting as a classical portfolio optimizer of terminal HARA utility with no funding constraint, $K_\nu(x) = x^\nu/\nu$, see (Merton, 1971). Constant equity allocation strategies are of less interest to our study, though, since insolvency becomes inevitable in our multi-period setting.

Note that equity shares larger than one correspond to leveraging, ie. borrowing money to invest in equity. Also notice that the bonus policy reduces the equity allocation at time points $t_1, t_2, \ldots t_n$, thereby potentially forcing the actual equity allocation far below the equity cap.

3. One Period Dynamics

In this section we will quantify the value to clients of the actions of the (management of the) pension company over one period. We will compute both the expected bonus level at the end of the period and the financial value of the bonus option. The analysis shows that for both measures the greatest value is created for the most aggressive strategies.

The prize for pursuing an aggressive strategy, however, is that it leaves the company with a low funding ratio with high probability and thereby at a worse starting point for the next period. The long-term effects of these dynamics on the bonus level and funding ratio distribution will be analyzed in the next section.

3.1. **Expected Bonus Level and Funding Ratio.** The level of bonus at the end of the period and the funding ratio after this bonus attribution are given by

$$r^B = (F_{T-} - \kappa)^+/\kappa, \quad F_T = \min\{F_{T-}, \kappa\}.$$

It follows from Theorem 2.1 after standard calculations that the expected value of these quantities become

$$\mathsf{E}[r^B] = \frac{f-1}{\kappa} e^{\tilde{\sigma}^2(1-\nu)T} N(d_1) - \frac{\kappa - 1}{\kappa} N(d_2),$$
$$\mathsf{E}[F_T] = 1 + (f-1)e^{\tilde{\sigma}^2(1-\nu)T}(1 - N(d_1)) + (\kappa - 1)N(d_2),$$

where $F_0 = f$, N is the cumulative normal distribution function, and

$$d_1 = \frac{1}{\sqrt{\tilde{\sigma}^2 T}} \left(\frac{\tilde{\sigma}^2}{2}(3 - 2\nu)T - \log \frac{\kappa - 1}{f - 1} \right), \quad d_2 = d_1 - \sqrt{\tilde{\sigma}^2 T}.$$

Fig. 1 shows an example of these expected values as a function of ν for $\kappa = 1.5$ and $F_0 = 1.25$. The expected bonus level diverges to infinity as ν approaches one, while the expected funding ratio after bonus tends to one in the same limit. Thus high equity allocations lead to high expected returns but also to low funding ratios with high probability.

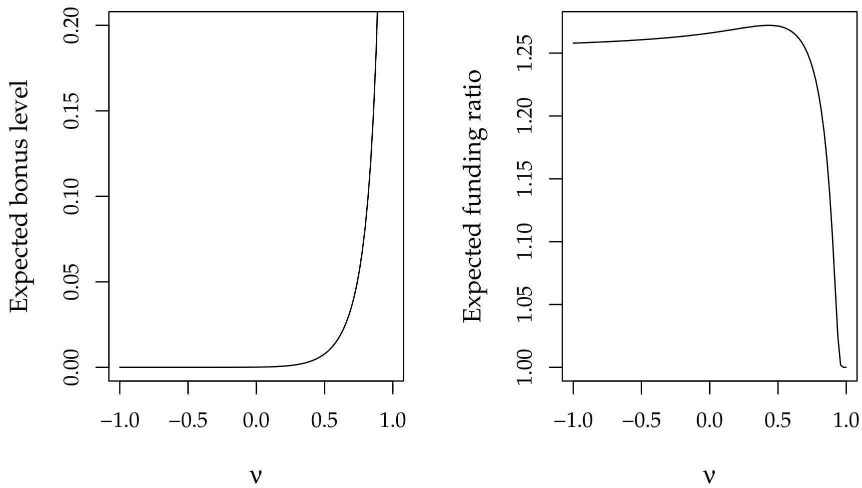

FIGURE 1. Expected bonus level and expected funding ratio after bonus attribution as functions of ν for $(r, \mu, \sigma, T, \kappa, F_0) = (0.04, 0.05, 0.2, 1, 1.5, 1.25)$.

3.2. **One Period Option Value.** From an option point of view the payout to a policy holder with a payment of 1 due at time T can be considered as an option consisting of a fixed payment of 1 and — potentially — an additional bonus if the funding ratio exceeds κ. The total payout, O, to be received by the policy holder at time T therefore is

$$O = 1 + (F_{T-} - \kappa)^+/\kappa.$$

The equivalent martingale measure, $\mathbb{P}^*$, is the measure under which

$$W_t^* = W_t + \frac{\mu}{\sigma}t \tag{10}$$

is Brownian motion, Björk (2004). In terms of W^* equity S_t, cf. (1), evolves according to

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^*, \tag{11}$$

and risk-neutrality is observed from the fact that $(e^{-rt}S_t)_{t\geq 0}$ is a $\mathbb{P}^*$-martingale.

The fair value of the option is the expected value of the discounted payout under the martingale measure $\mathbb{P}^*$. The (time 0) fair value of the option is thus given by

$$V_O = \mathsf{E}_{\mathbb{P}^*}\left[e^{-rT}O\right] = e^{-rT} + V_B,$$

where

$$V_B = \mathsf{E}_{\mathbb{P}^*}\left[e^{-rT}(F_{T-} - \kappa)^+/\kappa\right].$$

Under the risk-neutral measure, $\mathbb{P}^*$, the funding ratio therefore evolves like

$$F_t = (F_0 - 1)e^{-\frac{\tilde{\sigma}^2}{2}t + \tilde{\sigma}W_t^*} + 1, \tag{12}$$

where $\tilde{\sigma} = \mu/(\sigma(1-\nu))$. The result is non-trivial from two perspectives: First, volatility is inversely proportional to the underlying asset volatility, and, secondly, equity premium, μ, explicitly enters the equation of the risk-free dynamics.

It is well known from Black-Scholes option pricing theory that the value of an option written on S depends solely on the distribution of S under the equivalent martingale measure, $\mathbb{P}^*$. As this distribution does not depend on μ nor can the value of an option on S depend on μ.

The explanation to this apparent contradiction is that the bonus option is an option on the funding ratio process F, or equivalently the asset process A, rather than on S itself. Since the dynamics of A is the result of an optimization under the physical measure $\mathbb{P}$ the volatility of the process will depend on both the market price of risk (ie. the ratio μ/σ) and the risk aversion parameter ν. The result is that volatility and drift are tied together by the optimal investment strategy hence the explicit dependence on drift of option prices. This effect is also seen in (Teplá, 2001).

Regarding the value of the option standard calculations now give

$$V_B = \frac{e^{-rT}}{\kappa}\left[(f-1)N(d_1) - (\kappa - 1)N(d_2)\right] \tag{13}$$

where f is the initial funding ratio and

$$d_1 = \frac{1}{\sqrt{\tilde{\sigma}^2 T}}\left(\frac{\tilde{\sigma}^2}{2}T - \log\frac{\kappa - 1}{f - 1}\right), \quad d_2 = d_1 - \sqrt{\tilde{\sigma}^2 T}.$$

An example of the value of the bonus option as function of ν is shown in Fig. 2.

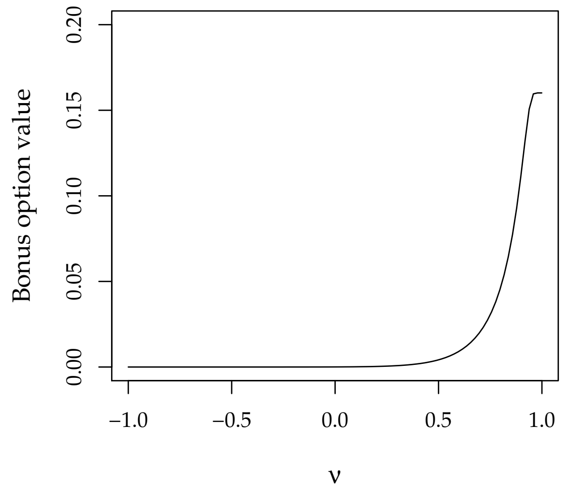

Figure 2. One-period value of bonus option as function of ν for $(r, \mu, \sigma, T, \kappa, F_0) = (0.04, 0.05, 0.2, 1, 1.5, 1.25)$.

Non-surprisingly, the value of the bonus option increases with the aggressiveness of the investment strategy. A risk-averse management simply creates less value to its customers — at least in the short run.

Another interesting question is: Where does the money come from? How can a newly committed T-dollar suddenly be worth more? For entrants to the pension scheme it really *is* a free lunch to participate in the bonus programme from day one. Under our assumption of zero net cash flow the "free" option given to a newly contributed dollar is handed over by pensioners receiving a dollar from the company thereby exactly forfeiting the bonus option on that dollar. In the general case where eg. the net cash flow is positive *all* current clients finance the bonus option of entrants since new entrants reduce overall funding ratio thereby reducing the value of the average bonus option. Similarly, the value of the average bonus increases when the net cash flow is negative. For a quantification of the transfer of wealth between generations in a collective pension scheme see (Døskeland and Nordahl, 2008).

The value of the bonus option for an investment strategy with a constant equity share is given in (Jørgensen, 2001).

4. Operating in Stationarity

The life-span of a with-profits contract is very long and it is therefore essential to analyze the multi-period behavior of the company based on the one-period optimization of section 2. For management, operating in stationarity is of essential job security but also clients have some interest although this is not completely clear.

In general non-stationarity can arise either from bankruptcy (or near-bankruptcy) of the company or divergence of the funding ratio to infinity as a result of no bonus attribution. From a client's perspective the latter case is the worst since the company simply keeps surplus returns in the investment collective to no benefit of clients. On the other hand, insolvency means the company is discontinued upon which the client will receive the guaranteed part of the contract ... and then simply go to another company. The route into insolvency could be generous bonus attribution, though, which would now be guaranteed and therefore included in the transfer to the new company. The problem in this case is the redistribution of wealth within the collective where new entrants receive heavily on a bonus option they never paid for.

In this section we will discuss the existence and characteristics of stationary versions of a company run ad infinitum. More specifically, we will study the stationary distribution of the funding ratio, measured immediately after annual bonus attribution, ie. $t_{i+1} - t_i = 1$, for a company in which the inflow and outflow of funds exactly match.

Under these assumptions we have that the yearly sampled funding ratio process, $F_i = F_{t_i}$, evolves as a discrete-time Markov chain with dynamics

$$F_i = \min\left\{(F_{i-1} - 1)e^{\frac{\tilde{\sigma}^2}{2}(1-2\nu)+\tilde{\sigma}U_i} + 1, \kappa\right\}, \tag{14}$$

where the U_i's are i.i.d. standard normal variables.

4.1. **Existence of Stationarity.** Clearly the funding ratio process F is restricted to lie between 1 and κ for all values of ν. This, however, does not imply that there exists a stationary distribution for all ν. In fact, we shall show below that too aggressive strategies, ie. too large values of ν, lead to non-stationary funding ratio processes which will drift to 1 in the long run.

The funding ratio process is most easily studied via the transformation

$$Y_i = -\log\left(\frac{F_i - 1}{\kappa - 1}\right),$$

which transforms the dynamics given in (14) to the simpler dynamics

$$Y_i = (Y_{i-1} + V_i)^+, \tag{15}$$

where the V_i's are i.i.d. normally distributed with mean $-\tilde{\sigma}^2(1-2\nu)/2$ and variance $\tilde{\sigma}^2$. The form (15) lends itself to investigation by Markov chain theory from which we get the following result

Theorem 4.1. *The funding ratio process in (14) admits a stationary distribution if and only if $\nu < 1/2$.*

Proof. It follows from Proposition 11.5.3 of Meyn and Tweedie (1993) that Y, and hence F, has a stationary distribution if and only if the mean of V_i is strictly negative, which is satisfied if and only if $\nu < 1/2$. □

Note that, surprisingly, the result does not depend on the capital market parameters μ and σ. Regardless of the values of these parameters the process Y wanders off to infinity for $\nu > 1/2$, while for $\nu = 1/2$ the process is so-called null-recurrent, ie. it returns infinitely many times to the origin but its distribution gets more and more diffuse. In both cases this implies that the funding ratio process — although strictly above 1 at all times — gets more and more concentrated around 1. Thus too aggressive investment strategies will eventually trap the company at near-bankruptcy.

4.2. **Stationary Distribution.** The one-sided random walk (15) has been the object of intense study in the queueing literature, but despite of this explicit expressions for the stationary distribution are generally unavailable.

Numerically, getting a sample from the stationary distribution is seemingly easy using either (14) or (15). However, for ν close to 1/2 convergence to stationarity is slow and it is hard to assess the number of time steps needed to (approximately) reach stationarity. A better approach is to use the fact that the stationary distribution of Y is equal to the distribution of

$$M = \max\{0, X_1, X_2, \ldots\}, \tag{16}$$

where $X_i = V_1 + \ldots + V_i$ is the underlying, unrestricted random walk, see e.g. Chapter XII of Feller (1971). Using this representation it is easy to sample from the stationary distribution within a given tolerance: Simply simulate the random walk X_i till it gets sufficiently negative and take the maximum of the path[1]. Simulated results using this method are shown in Fig. 3.

The representation (16) is also a fundamental tool in the theoretical study of the stationary distribution of Y.

[1]The level, b, needed to ensure a distribution with a total variation distance of at most p from the true stationary distribution can be determined as follows. Let X_t denote a Brownian motion with drift $-\tilde{\sigma}^2(1-2\nu)/2$ and variance $\tilde{\sigma}^2$, and define the stopping time $\tau_b = \inf\{t \geq s : X_t = 0\}$. It then follows from section 3.5.C of (Karatzas and Shreve, 1991) that $\mathbb{P}(\tau_b < \infty | X_s = b) = \exp((1-2\nu)b)$ for $b < 0$. Setting $b = \log(p)/(1-2\nu)$ thus ensures that once X_i gets below b the probability of it ever getting positive again is at most p. In the paper is used $p = 0.001$.

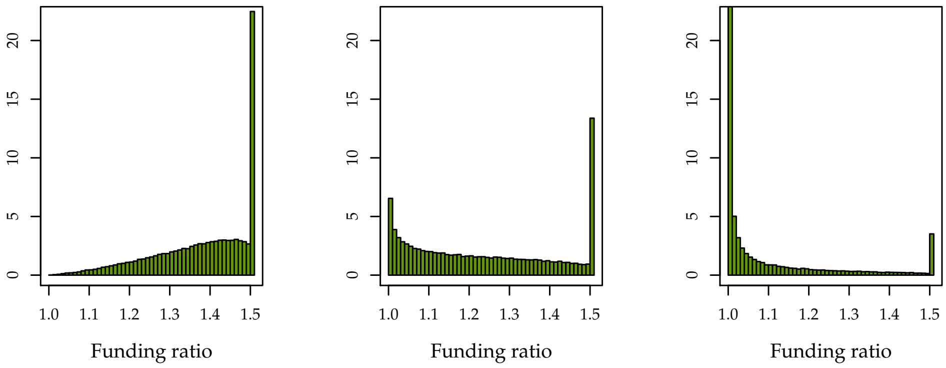

FIGURE 3. Simulated stationary distribution of funding ratios for $(r, \mu, \sigma, \kappa) = (0.04, 0.05, 0.2, 1.5)$ and $\nu = -2/3$ (left), $1/6$ (middle), and $4/9$ (right) corresponding to maximum equity holdings 25%, 50%, and 75% at κ. Based on 100.000 simulations.

Fig. 3 demonstrates that even when the company completely has removed insolvency risk an essential tradeoff between risk/reward remains. Risk, though, is now to be interpreted as stability of operation since risk of insolvency has manifestly been removed.

A risk-averse management targeting a low equity share (left panel, $\nu = -2/3$) will be successful in operating the pension company in a tight funding ratio regime at — and close to — the maximum attainable level, κ. The company will be able to attribute bonus at a high frequency since this is equivalent to the point probability mass at κ.

As investment risk is increased (middle panel, $\nu = 1/6$) the probability of attributing bonus decreases and — at the same time — a new risk emerges, that is, a risk of being temporarily trapped at a low funding ratio for a sustained period of time.

The risk of being trapped at a low funding ratio is even more pronounced for an aggressive investment strategy (right panel, $\nu = 4/9$) where the company spends most of its time at (very) low funding ratios and only rarely is able to attribute bonus.

The paradigm of investment theory is that more risk means a higher expected return. As we shall see in Sec. 4.3 this only partly holds true for with-profits contracts since higher risk, ie. a higher equity cap, as a side-effect results in a lower bonus frequency.

4.2.1. *Analytical Approximation.* A numerical simulation does not give much insight into the structure of the stationary distribution and we therefore develop an analytical approximation below. This is based on the special case where the tails of V_i are exponential in which case the stationary distribution of Y is known.

Specifically, assume V_i is of the form $A_i - B_i - c$, where A_i and B_i are independent and exponentially distributed both with mean $1/\lambda$ and c is a positive constant, then the distribution function of Y is given by

$$\mathbb{P}(Y \leq y) = 1 - \frac{\lambda - \rho}{\lambda} e^{-\rho y}, \tag{17}$$

where $0 < \rho < \lambda$ is the solution to the equation

$$\mathsf{E}(e^{\rho V_i}) = 1, \tag{18}$$

see Chapter XII of Feller (1971). Note that Y has a point mass of size ρ/λ at 0. Also note that c does not enter explicitly in the expression for the distribution function, but it is of course present through ρ.

We can use this result to obtain an approximation to the stationary distribution of Y in the case of interest where V_i is normally distributed. First, we determine λ to match the variance of $A_i - B_i - c$ to $\tilde{\sigma}^2$,

$$\mathrm{Var}(A_i - B_i - c) = \frac{2}{\lambda^2} = \tilde{\sigma}^2 \Leftrightarrow \lambda = \frac{\sqrt{2}}{\tilde{\sigma}}.$$

For the second step one possibility would be to let $c = \tilde{\sigma}^2(1 - 2\nu)/2$ and solve (18) with $V_i = A_i - B_i - c$ to determine ρ. This gives the equation

$$\frac{\lambda}{\lambda - \rho} \frac{\lambda}{\lambda + \rho} = e^{\rho c}. \tag{19}$$

However, since this approach does not yield an explicit expression for ρ we will instead solve (18) with the *original* normally distributed V_i,

$$\mathsf{E}(e^{\rho V_i}) = 1 \Leftrightarrow e^{-\rho\tilde{\sigma}^2(1-2\nu)/2 + \rho^2\tilde{\sigma}^2/2} = 1 \Leftrightarrow \rho = 1 - 2\nu.$$

Note that with this choice we are not guaranteed $\rho < \lambda$ for all ν unless $\sigma/\mu > \sqrt{2}$ (in case $\rho \geq \lambda$ one can use ρ given by (19) instead).

Transforming (17) back to the funding ratio scale we arrive at:

$$\begin{aligned}
\mathbb{P}(F \leq f) &= \mathbb{P}(Y \geq -\log[(f-1)/(\kappa-1)]) \\
&= 1 - \mathbb{P}(Y < -\log[(f-1)/(\kappa-1)]) \\
&\approx 1 - \left(1 - \frac{\lambda - \rho}{\lambda} e^{\rho \log \frac{f-1}{\kappa-1}} - \frac{\rho}{\lambda} 1_{\{\kappa\}}(f)\right) \\
&= \frac{\lambda - \rho}{\lambda} \left(\frac{f-1}{\kappa-1}\right)^{\rho} + \frac{\rho}{\lambda} 1_{\{\kappa\}}(f) \quad \text{for } 1 \leq f \leq \kappa.
\end{aligned} \tag{20}$$

Notice that the probability of giving bonus, ie. the point mass ρ/λ at κ, is *independent* of κ. The level at which bonus is attributed therefore does not affect how often bonus is attributed — only how much. The average bonus attributed therefore in general increase with κ. This point also applies to the true stationary distribution. Here the point mass is $\mathbb{P}(F = \kappa) = \mathbb{P}(Y = 0)$ which in general does not depend on κ since Y does not depend on κ.

Fig. 4 shows simulated and approximate cumulative distribution functions in stationarity of the examples in Fig. 3.

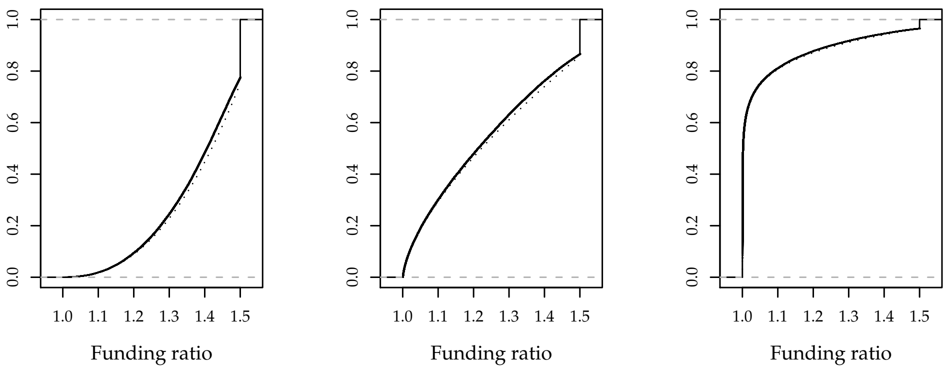

FIGURE 4. Cumulative distribution function of simulated (solid) and approximated (dashed) stationary funding ratios for $(r, \mu, \sigma, \kappa) = (0.04, 0.05, 0.2, 1.5)$ and $\nu = -2/3$ (left), 1/6 (middle), and 4/9 (right) corresponding to maximum equity holdings 25%, 50%, and 75% at κ. Simulated results based on 100.000 simulations.

From (20) it follows that the point mass at κ, ie. the probability of attributing bonus, is a decreasing function in ν. This is a paradox in pension fund management: the more aggressive the company invests the smaller the probability of actually giving a bonus.

Intuitively, the limiting case $\nu \to -\infty$ is relatively easy to understand: Assume the company does not invest in equity at all. In this case the funding ratio remains constant and the company never attributes bonus. Investing just the tiniest fraction in equity results in a positive drift upward in funding ratio which — eventually — will allow the company to attribute bonus. When stationarity is reached, bonus attribution will occur with the highest frequency possible, $\sqrt{2}\mu/\sigma$, and the funding ratio will stay within a very narrow range of κ. Obviously, bonuses given are very small.

Aggressive investment strategies, on the other hand, eventually force the company out of the stationary regime resulting in asymptotic insolvency of the company. In the transition out of stationarity, $\nu \to 1/2-$, the company will be successful in attributing high bonuses in some years but will in long periods be trapped at low funding ratios incapable of escape.

The management of a pension company therefore is faced with a difficult tradeoff between giving low bonuses at a high frequency or high bonuses at a low frequency. The value to clients of this choice is the subject of the next section.

4.3. **Expected Bonus Level and Funding Ratio.** A pension contract is usually composed of a series of premium payments over the working life of the insured – hopefully followed by many years in retirement. The time-span of the relation between a customer and the pension company therefore is very long. It is therefore reasonable to assume that over the life-time of a pension contract the realized return generated from the guaranteed rate and bonus effectively samples the distribution of bonus attribution which is stationary if the funding ratio of the company is stationary. If this assumption holds then the average return on an individual contract equals the average return of the investment collective as a whole.

As regards the expected funding ratio after bonus attribution in stationarity this is easily calculated from expression (20)

$$\mathsf{E}[F] \approx \int_1^\kappa f dG(f) = \kappa - \frac{\lambda-\rho}{\lambda}\frac{\kappa-1}{\rho+1}. \tag{21}$$

where G is the distribution function.

Calculating the bonus distribution in stationarity and its expectation is somewhat harder. The dynamics of the funding ratio process gives that the funding ratio immediately before bonus attribution in stationarity is distributed as

$$F_- = (F-1)e^{\frac{\tilde\sigma^2}{2}\rho + \tilde\sigma U} + 1,$$

where F follows the stationary distribution of the funding ratio after bonus attribution and U is a standard normal variate.

Assuming F is distributed according to G we can obtain an approximation to the distribution function of F_-

$$G_-(f) = \int_1^\kappa H(\tau)dG(\tau) = H(\kappa)G(\kappa) - \int_1^\kappa G(\tau)dH(\tau) \tag{22}$$

where

$$H(\tau) = \mathbb{P}(F_- \le f | F = \tau) = N\left(-\frac{\tilde\sigma\rho}{2} + \frac{\log\frac{f-1}{\tau-1}}{\tilde\sigma}\right).$$

The last term in (22) can be written

$$\begin{aligned}
&-\int_1^\kappa G(\tau)dH(\tau)\\
&= \frac{1}{\sqrt{2\pi}\tilde\sigma}\int_1^\kappa \frac{\lambda-\rho}{\lambda}\left(\frac{\tau-1}{\kappa-1}\right)^\rho e^{-\frac{1}{2}\left(-\frac{\tilde\sigma\rho}{2}+\frac{\log\frac{f-1}{\tau-1}}{\tilde\sigma}\right)^2}\frac{1}{\tau-1}d\tau\\
&= \frac{1}{\sqrt{2\pi}\tilde\sigma}\int_{-\infty}^{\log(\kappa-1)} \frac{\lambda-\rho}{\lambda}\left(\frac{f-1}{\kappa-1}\right)^\rho e^{-\frac{1}{2\tilde\sigma^2}\left(z-\frac{\tilde\sigma^2}{2}\rho-\log(f-1)\right)^2}dz\\
&= \frac{\lambda-\rho}{\lambda}\left(\frac{f-1}{\kappa-1}\right)^\rho N\left(-\frac{\tilde\sigma\rho}{2} - \frac{\log\frac{f-1}{\kappa-1}}{\tilde\sigma}\right),
\end{aligned}$$

and we thus have

$$G_-(f) = N(d_1) + \frac{\lambda - \rho}{\lambda}\left(\frac{f-1}{\kappa-1}\right)^{\rho} N(d_2) \quad \text{for } f \geq 1, \tag{23}$$

where

$$d_1 = -\frac{\tilde{\sigma}\rho}{2} + \frac{\log\frac{f-1}{\kappa-1}}{\tilde{\sigma}}, \quad d_2 = -d_1 - \rho\tilde{\sigma}.$$

Note, that we now have two approximations to the stationary probability of attributing bonus, $1 - G_-(\kappa)$ and $G(\{\kappa\}) = \rho/\lambda$. Arguably F_- is "one step closer to stationarity" than F and $1 - G_-(\kappa)$ therefore the better approximation.

The bonus level then is approximately distributed as the scaled tail of G_-

$$\mathbb{P}(r^B \leq r) = \mathbb{P}((F_- - \kappa)/\kappa \leq r) \approx G_-(\kappa(1+r)) \quad \text{for } r \geq 0.$$

Fig. 5 shows approximate and simulated densities of bonus levels. The higher the allocation to equity the higher bonus levels become but tails also become thinner since the probability of *not* attributing grows as well. Point masses of zero bonus are therefore indicated as well. A given year's apparent success of a large bonus resulting from a high equity allocation therefore can come at the even higher price of a large loss the next year trapping the company for a long period at a low funding ratio.

The tails of Fig. 5 clearly illustrates the point made several times by now, that is, that the more aggressive the pension company invests the larger the bonus becomes . . . but also the rarer.

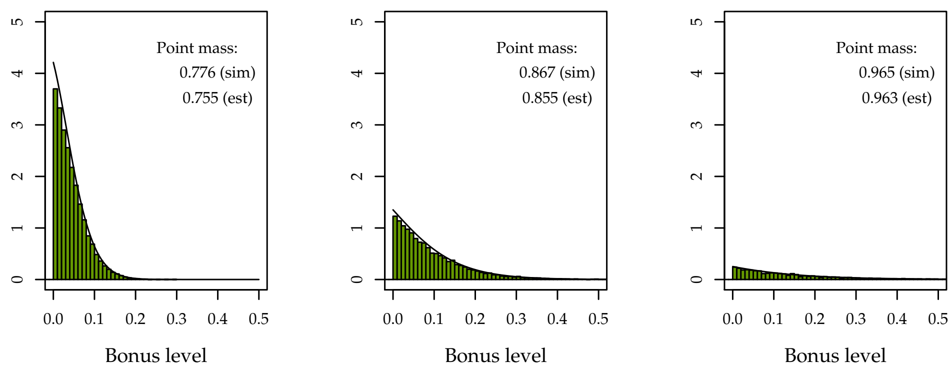

FIGURE 5. Distribution of bonus. Density of simulated (histogram) and approximated (line) bonus attribution ($r^B > 0$) for $(r, \mu, \sigma, \kappa) = (0.04, 0.05, 0.2, 1.5)$ and $\nu = -2/3$ (left), $1/6$ (middle), and $4/9$ (right) corresponding to maximum equity holdings 25%, 50%, and 75% at κ. Point mass for $r^B = 0$ indicated in figure. Based on 100.000 simulations.

Given the approximate distribution function of F_- we can derive an approximation to the expected bonus level in stationarity

$$
\begin{aligned}
\mathsf{E}(r^B) &= \mathsf{E}\left(\frac{F_- - \kappa}{\kappa} 1_{(F_- > \kappa)}\right) \\
&= \frac{1}{\kappa}\left(\mathsf{E}(F_- 1_{(F_- > \kappa)}) - \kappa \mathbb{P}(F_- > \kappa)\right) \\
&\approx \frac{1}{\kappa}\left(\int_\kappa^\infty f dG_-(f) - \kappa(1 - G_-(\kappa))\right) \\
&= \frac{1}{\kappa}\left(\int_\kappa^\infty (f-1) dG_-(f) + (1-\kappa)(1 - G_-(\kappa))\right),
\end{aligned}
$$

where

$$
\frac{dG_-(f)}{df} = \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}\tilde{\sigma}(f-1)} + \frac{\lambda - \rho}{\lambda}\left(\frac{f-1}{\kappa-1}\right)^\rho \left(\frac{\rho}{f-1} N(d_2) - \frac{e^{-\frac{1}{2}d_2^2}}{\sqrt{2\pi}\tilde{\sigma}(f-1)}\right).
$$

After lengthy, but tedious, calculations similar to those leading to G_- we arrive at

$$
\mathsf{E}(r^B) \approx \frac{\kappa - 1}{\kappa}\left(\frac{\rho(\lambda+1)}{\lambda(\rho+1)} e^{\frac{\tilde{\sigma}^2}{2}(\rho+1)} N(\frac{\tilde{\sigma}\rho}{2} + \tilde{\sigma}) + \frac{\lambda(\rho+2) - \rho}{\lambda(\rho+1)} N(-\frac{\tilde{\sigma}\rho}{2}) - 1\right).
$$

The expected bonus level and the expected funding ratio after bonus attribution in stationarity as functions of ν are shown in Fig. 6. The expected bonus level attains its maximum when management shows moderate restraint on aggressiveness and chooses $\nu = 0.063$ corresponding to an equity allocation of 44% at κ. Comparing with Fig. 1 for the one-period case, the expected bonus level in both cases tend to zero for $\nu \to -\infty$. However, in contrast to the one-period case the stationary bonus level also tends to zero for $\nu \to 1/2-$. This is caused by the deterioration of the funding ratio as shown in the right panel of Fig. 6.

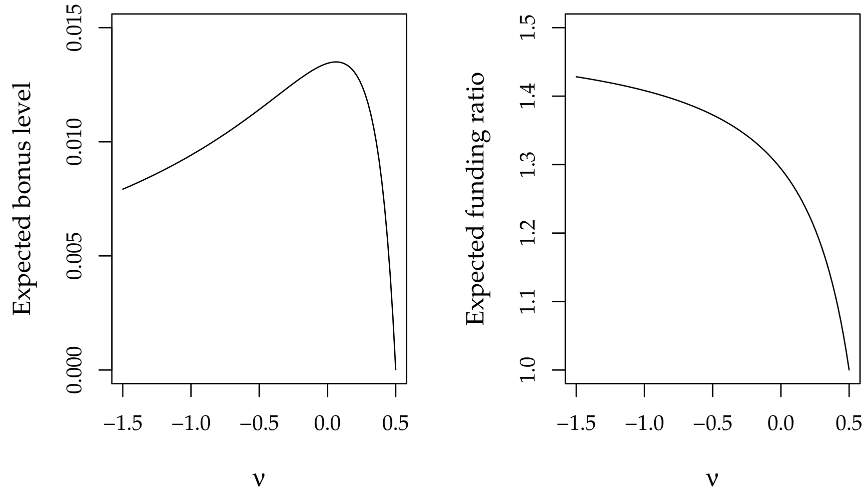

FIGURE 6. Approximate expected bonus level and expected funding ratio after bonus attribution in stationarity as functions of ν for $(r, \mu, \sigma, \kappa) = (0.04, 0.05, 0.2, 1.5)$. Maximum expected bonus of 1.35% is attained at $\nu = 0.063$.

For investment strategies derived from high risk-aversion, ie. low ν, the paradigm of classic investment theory still holds and expected return increases with risk. But a maximum to expected bonus must exist since the equity holding goes to zero for $\nu \to -\infty$ and the probability of receiving a bonus goes to zero as $\nu \to 1/2-$ which in both cases lead to an expected bonus level of zero. Thus in general the maximum stationary bonus level must be attained for a risk-aversion level strictly between $-\infty$ and $1/2$.

The analysis above applies to a single bonus attribution in the future (stationarity). However, since a typical pension contract runs for many years it is also of considerable interest to study the accumulated effect of a number of consecutive bonus attributions. This extension involves a detailed study of the correlation structure of bonus attributions over time and is carried out in (Kryger, 2008). Approximations to the terminal value distribution of a contract receiving smoothed annual returns are given in (Jørgensen, 2007).

5. Conclusion

We have investigated the effects of investing for retirement through a with-profits pension scheme. We have assumed that management of the pension company acts as a one-period optimizer according to a terminal utility that quantifies preferences for risk versus returns. The utility of terminal wealth can be thought of as weighting the commercial and competitive advantages of high returns against the risks of low returns ... for the company.

Clients of with-profits pension schemes are *not* one-period optimizers but participate in the scheme for many consecutive periods. What seems to be a series of sound, optimal short-term investment decisions is therefore not a sufficient condition to ensure long-term stability. This is because previous years' losses directly depress coming years' returns - potentially for a very long time.

To clients — and management — operating the company in stationarity therefore is of crucial importance: to clients lack of stationarity either means bankruptcy or near-bankruptcy of the company or divergence in funding ratio both resulting in loss of expected bonus; to management loss of status and personal income. We showed that the existence of stationarity effectively is a restriction to the class of utilities considered.

Given the company is operated in stationarity the excess return to clients in our model samples the stationary distribution of the funding ratio which results also in stationarity of bonus levels. We showed that expected bonus level must attain a maximum value within the range of valid (stationary) utilities because expected bonus towards both end-points of this range is zero.

In conclusion the optimal performance of a pension contract from a client's perspective is therefore not attained for the most aggressive (one-period) investment strategy allowed within the stationary regime but at a more moderate investment strategy. A company's incentive to boast high (period) returns therefore can be in conflict with the best interests of its clients.

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The Danish Labour Market Supplementary Pension Scheme, Kongens Vænge 8, DK-3400 Hillerød, Denmark