Some solvable portfolio problems with quadratic and collective objectives

Esben Masotti Kryger and Mogens Steffensen

March 23, 2010

Abstract

We present a verification result for a general class of portfolio problems, where the standard dynamic programming principle does not hold. Explicit solutions to a series of cases are provided. They include dynamic mean-standard deviation, endogenous habit formation for quadratic utility, and group utility. The latter is defined by adding up the certainty equivalents of the group members, and the problem is solved for exponential and power utility.

1 Introduction

For decades the class of HJB-solvable dynamic asset allocation problems over terminal wealth, $X(T)$, has been limited to those in the form

$$\sup_{\pi} E_{t,x}\left[F\left(X(T)\right)\right],$$

for some function F. Björk and Murgoci (2008) extended this class to those in the form

$$\sup_{\pi}\left[E_{t,x}\left[F\left(t,x,X(T)\right)\right]+G\left(t,x,E_{t,x}\left[X(T)\right]\right)\right], \tag{1}$$

for some function G, which allowed them to calculate the optimal *time consistent* investment strategy for a mean-variance investor. This result was first published by Basak and Chabakauri (2009) in a quite general incomplete market framework. The novelty of Björk and Murgoci (2008) is, apart from working in a general Markovian financial market, the dependence on (t,x) in their F as well as the mere presence of a G that is not affine in wealth. Furthermore they allow for consumption, skipped in (1). The dependence on (t,x) and the non-affine G rule out the use of the classical Bellmann-technique, and, consequently, they refer to such problems as *time inconsistent*. Equivalently, the definition in Basak and Chabakauri (2009) is "policies, from which the investor has [an] incentive to deviate".

The aim of the present paper is to study the class of problems in the form

$$\sup_{\pi} f\left(t,x,E_{t,x}\left[g_1\left(X(T)\right],\ldots,g_n\left(X(T)\right)\right]\right),$$

where f is allowed to be non-affine in the g-functions. Our main application is a group utility problem, where a group of investors seek to maximize a specific notion of group utility, where investors share terminal wealth equally. Whereas utility maximization for a single investor may be considered a classic problem it is not clear how to formalize the preferences of a group of heterogeneous agents. We suggest to maximize the sum of certainty equivalents, and thereby form the objective

$$\sum_{i=1}^{n} u_i^{-1}\left(E_{t,x}\left[u_i\left(X^{\pi}\left(T\right)/n\right)\right]\right),$$

for individual utility functions $u_1,\ldots,u_n$. Note that due to monotonicity of u_1 this problem is equivalent to the standard problem in the single investor case.

A different problem of interest that is contained in our general objective is mean-standard deviation optimization.

Both problems call on our general objective, and are not special cases of Björk and Murgoci (2008). On the other hand, due to the presence of t, x in their F-function they can deal with problems that we cannot treat.

The concept of time-inconsistency was first treated formally by Strotz (1956), who considered a so-called "Cake-Eating Problem" (i.e. one of allocating an endowment between different points in time). He showed that the optimal solution is time consistent only for exponential discounting. Strotz (1956) described three different types of agents, and Pollak (1968) contributed further to the understanding and naming of them: 1) the pre-committed agent does not revise his initially decided strategy even if that makes his strategy time-inconsistent; 2) the naive agent revises his strategy without taking future revisions into account even if that makes his strategy time-inconsistent; 3) the sophisticated agent revises his strategy taking possible future revisions into account, and by avoiding such makes his strategy time-consistent. Which type is more relevant depends on the entire framework of the decision in question. Here, we focus on the pre-committed and sophisticated agents and pay no attention to the naive agent. Strotz (1956) suggests that, although (in some sense) optimal, it may be difficult to pre-commit.

In recent years the concept of non-exponential (e.g. hyperbolic) discounting has received a lot of attention as a prime example of a time inconsistent problem. Solano and Navas (2009) give an overview over which strategies the three different types of agents should use.

A theorem, which characterizes the solution (in a Black-Scholes market) to our class of problems is provided in Section 2, while Sections 3 and 4 present applications, some of which are - to our knowledge - new. Finally, Section 5 wraps the findings up and provides an outlook on further work within this area.

2 The Main Result

We consider a market consisting of a bond and a stock with dynamics given by

$$\begin{aligned} dB(t) &= rB(t)\,dt, B(0) = 1 \\ dS(t) &= \alpha S(t)\,dt + \sigma S(t)\,dW(t), S(0) = s_0 > 0, \end{aligned}$$

with $r < \alpha$ and $\alpha,\ \sigma > 0$. W is a standard Brownian motion on an abstract probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions; and with each $\mathcal{F}_t \subseteq \sigma\{W(s),\ 0 \leq s \leq t\}$. Further we define $\theta = (\alpha - r)/\sigma$, the market price of risk.

We consider an investor, who places the proportion $\pi(t)$ of his wealth in the stock at time t. Denoting by $X^\pi(t)$ his wealth at time t given the investment strategy π, the dynamics of his wealth becomes

$$\begin{aligned} dX^\pi(t) &= (r + \pi(t)(\alpha - r)) X^\pi(t)\,dt + \pi(t)\sigma X^\pi(t)\,dW(t), \\ X^\pi(0) &= x_0 > 0, \end{aligned} \tag{2}$$

where x_0 is the initial wealth. The strategy is self-financing in the sense that we disregard consumption and injection of capital.

Before introducing the objectives we introduce two conditional expectations

$$\begin{aligned} y^\pi(t,x) &= E_{t,x}[g(X^\pi(T))], \\ z^\pi(t,x) &= E_{t,x}[h(X^\pi(T))], \end{aligned}$$

for functions g and h. The subscript t, x denotes conditioning on the event $X^\pi(t) = x$.

The objective of the investor is to find

$$V(t,x) = \sup_\pi V^\pi(t,x) = \sup_\pi f(t, x, y^\pi(t,x), z^\pi(t,x)), \tag{3}$$

for a given regular function $f \in C^{1,2,2,2}$, and to find the corresponding optimal investment strategy, π^*.

As opposed to Björk and Murgoci (2008) we only treat problems over terminal wealth. Also, we restrict ourselves to (one-dimensional) Black-Scholes markets.

The portfolio problem presented in (3) is, in general, not a classical portfolio problem. If f does not depend on $(t, x, z^{\pi}(t,x))$ and is affine in $y^{\pi}(t,x)$, the problem can be written in a classical way,

$$V(t,x) \propto \sup_{\pi} E_{t,x}\left[g\left(X^{\pi}(T)\right)\right] + \text{ constant.} \tag{4}$$

The problem (3) is, at first glance, just a mathematical abstract generalization of the problem (4). However, as we argue below, there are examples of this generalization that make good economic sense. Truly, there are also examples of (3) that make no economic sense. But this is not an argument against solving (3) in its generality, as long as we have some interesting and useful applications in mind. Here we present a list of five such examples, which will be solved for in Sections 3 and 4. The first four examples are based on the specifications $g(x) = x$ and $h(x) = x^2$. In the fifth example, g and h are utility functions, and so-called group utility is maximized.

1. Mean-variance optimization with pre-commitment

   This is a classical quadratic utility optimization problem corresponding to

   $$f(t,x,y,z) = ay + bz + c \tag{5}$$

   When studying this example in detail in Subsection 3.4, we explain how this choice of f can deal with both mean-variance utility maximization and variance minimization under minimum return constraints. Essentially, we do not need the generalization (3) for this problem. For an appropriate choice of the function g, this is of course also a special case of (4). This relates to the fact that f does not depend on (t,x) and is both additive in (y,z) and linear in both y and z.

2. Mean-variance optimization without pre-commitment

   $$f(t,x,y,z) = y - \frac{\gamma(t,x)}{2}\left(z - y^2\right)$$

   If γ does not depend on t, x, f does not depend on (t,x), and there is additivity in (y,z) and linearity in z. But the non-linearity in y makes the problem non-standard. For γ constant, this is the problem treated by Basak and Chabakauri (2009) in an incomplete market framework. It is studied as a special (the simplest) case by Björk and Murgoci (2008). The case of $\gamma(x) = \gamma/x$ is investigated by Björk et al. (2009).

3. Mean-standard deviation optimization

   $$f(t,x,y,z) = y - \gamma\left(z - y^2\right)^{\frac{1}{2}}$$

   Due to the non-additivity of f in y and z, this case is not covered by Björk and Murgoci (2008)).

4. Quadratic utility with endogenous habit formation

   $$f(t,x,y,z) = -\left(\frac{1}{2}z + \frac{1}{2}x^2h^2(t) - xh(t)y\right).$$

   We provide the full solution to this problem although the case can also be solved using Björk and Murgoci (2008).

5. Collective of heterogeneous investors

   $$f(t,x,y,z) = g^{-1}(y) + h^{-1}(z),$$

where g and h are utility functions. E.g. for power utility

$$g(x) = x^{\gamma_1},$$
$$h(x) = x^{\gamma_2},$$
$$f(t, x, y, z) = y^{\gamma_1^{-1}} + z^{\gamma_2^{-1}}.$$

The functions g and h form the utility of terminal wealth, whereas the function f adds up the so-called certainty equivalents of the two investors. Whereas it may make no economic sense to add up the indirect utility from each investor (e.g. that would add up currency unit in different power), it makes good economic sense to add up certainty equivalents (at least that would add up linear currency units). However, the transition into certainty equivalents before adding up makes the problem non-standard due to the non-linearity of g^{-1} and h^{-1}. In order to extend to more than two agents, f needs more arguments, of course. To our knowledge this problem is new.

One can come up with several other interesting examples, but these are the ones we shall study in the present paper.

The results that facilitates the solution of this new class of problems is the following theorem.

Theorem 1 *Let $f : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ be a function from $C^{1,2,2,2}$. Let g and h be real functions. The set of* admissible *strategies are those, for which the stochastic integrals in (49) and (54) are martingales, and for which the partial differential equations (46)-(47) and (50)-(51) have solutions. Note that admissibility depends on the choice of g, h.*

Define $V(t, x) = \sup_\pi f(t, x, y^\pi(t, x), z^\pi(t, x))$ with the supremum taken over all admissible strategies, and with

$$y^\pi(t, x) = E_{t,x}[g(X^\pi(T))],$$
$$z^\pi(t, x) = E_{t,x}[h(X^\pi(T))].$$

If there exist three functions F, G, H such that

$$F_t - f_t = \inf_\pi \left[-(r + \pi(\alpha - r)) x (F_x - f_x) - \frac{1}{2}\sigma^2\pi^2x^2(F_{xx} - U) \right], \tag{6}$$

$$F(T, x) = f(T, x, g(x), h(x)), \tag{7}$$

$$G_t = -(r + \pi^*(\alpha - r)) x G_x - \frac{1}{2}\sigma^2(\pi^*)^2 x^2 G_{xx}, \tag{8}$$

$$G(T, x) = g(x),$$

$$H_t = -(r + \pi^*(\alpha - r)) x H_x - \frac{1}{2}\sigma^2(\pi^*)^2 x^2 H_{xx}, \tag{9}$$

$$H(T, x) = h(x),$$

where

$$U(f, y, z) = f_{xx} + 2f_{xy}y_x + 2f_{xz}z_x + f_{yy}y_x^2 + 2f_{yz}y_xz_x + f_{zz}z_x^2, \tag{10}$$

and

$$\pi^* = \arg\inf_\pi \left[-(r + \pi(\alpha - r)) x (F_x - f_x) - \frac{1}{2}\sigma^2\pi^2x^2(F_{xx} - U) + f_t \right]$$

Then

$$V(t, x) = F(t, x), y^{\pi^*}(t, x) = G(t, x), z^{\pi^*}(t, x) = H(t, x),$$

and the optimal investment strategy is given by π^*.

Proof. See Appendix. ■

We find the optimizing investment strategy in terms of the value function by differentiating with respect to π inside the square brackets of (6) and get

$$\pi^* = -\frac{\alpha - r}{\sigma^2 x} \frac{F_x - f_x}{F_{xx} - U(f, y, z)} \tag{11}$$

(provided $U > F_{xx}$). Feeding this control process back into the Bellman-like equation we get the following system of PDEs that we need to solve:

$$
\begin{aligned}
F_t &= -rx\left(F_x - f_x\right) + \frac{1}{2}\theta^2 \frac{\left(F_x - f_x\right)^2}{F_{xx} - U\left(f, G, H\right)} + f_t, && (12)\\
F\left(T, x\right) &= f\left(T, x, g\left(x\right), h\left(x\right)\right), \\
G_t &= -\left(rx - \theta^2 \frac{F_x - f_x}{F_{xx} - U\left(f, G, H\right)}\right) G_x - \frac{1}{2}\theta^2 \left(\frac{F_x - f_x}{F_{xx} - U\left(f, G, H\right)}\right)^2 G_{xx}, && (13)\\
G\left(T, x\right) &= g\left(x\right), \\
H_t &= -\left(rx - \theta^2 \frac{F_x - f_x}{F_{xx} - U\left(f, G, H\right)}\right) H_x - \frac{1}{2}\theta^2 \left(\frac{F_x - f_x}{F_{xx} - U\left(f, G, H\right)}\right)^2 H_{xx}, && (14)\\
H\left(T, x\right) &= h\left(x\right).
\end{aligned}
$$

We also present the system in terms of π^*, since this is sometimes convenient to work with:

$$
\begin{aligned}
F_t &= -rx\left(F_x - f_x\right) - \frac{1}{2}\left(\alpha - r\right)\pi^*\left(F_x - f_x\right)x + f_t, && (15)\\
G_t &= -\left(r + \pi^*\left(\alpha - r\right)\right)xG_x - \frac{1}{2}\sigma^2\left(\pi^*\right)^2 x^2 G_{xx}, && (16)\\
H_t &= -\left(r + \pi^*\left(\alpha - r\right)\right)xH_x - \frac{1}{2}\sigma^2\left(\pi^*\right)^2 x^2 H_{xx}, && (17)
\end{aligned}
$$

(with unchanged boundary conditions).

Remark 2 *The theorem can easily be extended to cover more than two transformations of terminal wealth:*

$$V(t,x) = \sup_{\pi} f\left(t, x, y_1^{\pi}(t,x), \ldots, y_n^{\pi}(t,x)\right),$$

in which case

$$U(f, y_1, \ldots, y_n) = \frac{\partial^2 f}{\partial x^2} + 2\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x y_i}\frac{\partial y_i}{\partial x} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial y_i y_j}\frac{\partial y_i}{\partial x}\frac{\partial y_j}{\partial x},$$

for $y_i^{\pi} = E_{t,x}\left[g_i\left(X^{\pi}\left(T\right)\right)\right]$.

Remark 3 *The standard case can be formalized by*

$$f\left(t, x, y^{\pi}\left(t,x\right), z^{\pi}\left(t,x\right)\right) = y^{\pi}\left(t,x\right)$$

Then the result collapses into a standard Bellman equation. This is seen by realizing that

$$f_t = f_x = U = 0. \qquad (18)$$

In this case $F = G$ and the differential equation for G (and also for H of course, since f does not depend on z) is redundant in the Theorem.

Also the proof collapses into a standard proof for the Bellman equation.

In the next two sections we solve the five problems listed above and variations thereof.

3 Quadratic Objectives

This section analyses the first four problems from the list in Section 2, albeit in a different order.

3.1 Mean-Variance without pre-commitment

In this subsection we consider the optimization problem

$$V(t,x) = \sup_{\pi} \left(E_{t,x}\left[X^{\pi}(T)\right] - \frac{\gamma(t,x)}{2} Var_{t,x}\left[X^{\pi}(T)\right] \right).$$

When $\gamma(t,x) = \gamma$ the solution to this problem was found by Basak and Chabakauri (2009) in a relatively general incomplete market. Björk and Murgoci (2008) also give the solution as *the* example of their rather general method.

For constant γ the function f is given by

$$f(t,x,y,z) = y - \frac{\gamma}{2}\left(z - y^2\right), \tag{19}$$
$$f_y = 1 + \gamma y, f_{yy} = \gamma, f_z = -\frac{\gamma}{2},$$
$$f_t = f_x = f_{xx} = f_{zz} = f_{xy} = f_{xz} = f_{yz} = 0.$$

From (10) we can now derive

$$U = \gamma G_x^2.$$

Plugging $f_t = f_x = 0$ and U into (11) and (12) we get the following optimal investment candidate and PDE that we need to solve together with (16),

$$\pi^* = -\frac{\alpha - r}{\sigma^2 x} \frac{F_x}{F_{xx} - \gamma G_x^2} \tag{20}$$
$$F_t = -rxF_x + \frac{1}{2}\theta^2 \frac{F_x^2}{F_{xx} - \gamma G_x^2}, \tag{21}$$
$$F(T,x) = x.$$

In this particular case the PDE for F involves G but not H and therefore we do not need to pay attention to the PDE for H. After having derived the solution to (21), this is plugged into (20) to form the optimal investment strategy as a function of (t,x). Plugging this strategy into (9) results in a PDE characterizing H. However we do not need this characterization in order to find the optimal investment and the value function.

We now search for a solution in the form

$$F(t,x) = p(t)x + q(t), G(t,x) = a(t)x + b(t).$$

Note that, for such a solution we can immediately derive from (19) that

$$H(t,x) = (ax+b)^2 + \frac{2}{\gamma}(ax + b - px - q).$$

The partial derivatives are

$$F_t = p'(t)x + q'(t), F_x = p(t), F_{xx} = 0,$$
$$G_t = a'(t)x + b'(t), G_x = a(t), G_{xx} = 0,$$

such that the optimal investment candidate (20) becomes

$$\pi^* = \frac{\alpha - r}{\gamma\sigma^2 x} \frac{p(t)}{a^2(t)}.$$

Plugging this strategy and the partial derivatives into (21) and (16) gives the system

$$p'(t)x + q'(t) = -rxp(t) - \frac{1}{2}\theta^2 \frac{p(t)^2}{\gamma a(t)^2},$$
$$a'(t)x + b'(t) = -rxa(t) - \theta^2 \frac{p(t)}{\gamma a(t)}.$$

Collecting terms with and without x gives

$$\begin{aligned}
p'(t) &= -rp(t), p(T) = 1, \\
q'(t) &= -\frac{1}{2}\theta^2 \frac{p(t)^2}{\gamma a(t)^2}, q(T) = 0, \\
a'(t) &= -ra(t), a(T) = 1, \\
b'(t) &= -\frac{\theta^2}{\gamma}\frac{p(t)}{a(t)}, b(T) = 0,
\end{aligned}$$

with solutions

$$\begin{aligned}
p(t) &= e^{r(T-t)}, \\
a(t) &= e^{r(T-t)}.
\end{aligned}$$

Also,

$$q(t) = b(t)/2 = \frac{\theta^2}{2\gamma}(T-t).$$

The optimal investment strategy finally becomes

$$\pi^*(t,x)x = \frac{\alpha - r}{\gamma\sigma^2}e^{-r(T-t)}.$$

This verifies the result of Basak and Chabakauri (2009) and Björk and Murgoci (2008).

The optimal strategy consists of putting a nominally increasing dollar amount in the risky asset - on most paths corresponding to a decreasing relative allocation. The discounted certainty equivalent is $e^{-r(T-t)}G(t,x) = x + \pi^*(t,x)x(\alpha - r)(T-t)$. This quantity has the trivial interpretation that the "value" of acting optimally in the market is exactly what could be obtained by investing the currently optimal dollar amount and harvesting the risk premium thereof with certainty. This is a bit special and results from the property that the certainty equivalent of the mean-variance object is merely the mean itself.

A constant γ is not an obvious model choice in that this penalty parameter must necessarily be estimated from the time-0 distribution of terminal wealth, which in turn depends on time to maturity (and thus calendar time) as well as present wealth. Therefore it could also be updated dynamically as the terminal wealth distribution changes as a result of market dynamics (and deterministically changing time to maturity). That is, γ could depend on x, and possibly on t. In the case treated above the agent pre-commits to γ but not to the target in the quadratic deviation forming the variance, cf. Subsection 3.4. Subsection 3.4 describes the classical case with pre-commitment to both quantities. Within the framework of the present subsection Björk et al. (2009) found a solution for the special case $\gamma(x) = \gamma/x$, where the investor does not pre-commit to any of the two.

3.2 Mean-Standard Deviation

Inspired by the discussion in the preceding subsection it is natural to modify the problem, seemingly slightly, to penalize with standard deviation instead of variance. In single-period models it is well-known that mean-variance and mean-standard deviation are equivalent - in the sense that the set of risk aversions maps into the same set of controls. As it turns out, this equivalence does not carry over to the dynamic model.

The optimization problem considered in this subsection is thus

$$\sup_{\pi}\left(E_{t,x}[X(T)] - \gamma\left(Var_{t,x}[X(T)]\right)^{\frac{1}{2}}\right).$$

To our knowledge this problem has not been studied before, but our extension of Björk and Murgoci (2008) makes it open to investigation.

The problem corresponds to the function f given by

$$f(t,x,y,z) = y - \gamma\left(z - y^2\right)^{\frac{1}{2}}, \tag{22}$$

$$\begin{aligned}
f_t &= f_x = f_{xx} = f_{xy} = f_{xz} = 0,\\
f_y &= 1 + y\gamma\left(z - y^2\right)^{-\frac{1}{2}},\\
f_{yy} &= \gamma\left(z - y^2\right)^{-\frac{1}{2}} + y^2\gamma\left(z - y^2\right)^{-\frac{3}{2}},\\
&= \gamma z(z-y^2)^{-3/2},\\
f_z &= -\frac{1}{2}\gamma\left(z - y^2\right)^{-\frac{1}{2}},\\
f_{zz} &= \frac{1}{4}\gamma\left(z - y^2\right)^{-\frac{3}{2}},\\
f_{yz} &= -\frac{1}{2}y\gamma\left(z - y^2\right)^{-\frac{3}{2}}.
\end{aligned}$$

From (10) we can now derive

$$\begin{aligned}
U &= \frac{1}{4}\gamma\left(H - G^2\right)^{-\frac{3}{2}} H_x^2 + \left(\gamma\left(H - G^2\right)^{-\frac{1}{2}} + G^2\gamma\left(H - G^2\right)^{-\frac{3}{2}}\right) G_x^2\\
&- G\gamma\left(H - G^2\right)^{-\frac{3}{2}} G_x H_x\\
&= \gamma\left(H - G^2\right)^{-\frac{1}{2}}\left(\frac{1}{4}\left(H - G^2\right)^{-1}\left(H_x - 2GG_x\right)^2 + G_x^2\right)\\
&= \gamma(H-G^2)^{-3/2}\left(HG_x^2 - GG_xH_x + H_x^2/4\right).
\end{aligned}$$

Plugging $f_t = f_x = 0$ and U into (11) and (12) we get the following optimal investment candidate and PDE that we need to solve together with (16) and (17),

$$\begin{aligned}
\pi^* &= -\frac{\alpha - r}{\sigma^2 x}\frac{F_x}{F_{xx} - \gamma\left(H - G^2\right)^{-\frac{1}{2}}\left(\frac{1}{4}\left(H - G^2\right)^{-1}\left(H_x - 2GG_x\right)^2 + G_x^2\right)}, \qquad (23)\\
F_t &= -rxF_x + \frac{1}{2}\theta^2\frac{F_x^2}{F_{xx} - \gamma\left(H - G^2\right)^{-\frac{1}{2}}\left(\frac{1}{4}\left(H - G^2\right)^{-1}\left(H_x - 2GG_x\right)^2 + G_x^2\right)},\\
F(T,x) &= x.
\end{aligned}$$

We now search for a solution in the form

$$F(t,x) = p(t)\,x, G(t,x) = a(t)\,x, H(t,x) = c(t)\,x^2,$$

with $c \geq a^2$. We know immediately from (22) that the following relation must hold

$$p(t) = a(t) - \gamma\left(c(t) - a^2(t)\right)^{\frac{1}{2}}.$$

The partial derivatives are

$$\begin{aligned}
F_t &= p'(t)\,x, F_x = p(t), F_{xx} = 0,\\
G_t &= a'(t)\,x, G_x = a(t), G_{xx} = 0,\\
H_t &= c'(t)\,x^2, H_x = 2c(t)\,x, H_{xx} = 2c(t),
\end{aligned}$$

such that the function U and optimal investment candidate (23) becomes

$$\begin{aligned}
U(t,x) &= \frac{\gamma\left(c(t) - a^2(t)\right)^{-\frac{1}{2}} c(t)}{x},\\
\pi^* &= \frac{\alpha - r}{\gamma\sigma^2}\frac{p(t)}{\left(c(t) - a^2(t)\right)^{-\frac{1}{2}} c(t)}.
\end{aligned}$$

Plugging this strategy and the partial derivatives into (15), (16), and (17) (and (12), (13), and (14) for the boundary conditions) gives the system

$$\begin{aligned}
p' &= -\left(r + \frac{1}{2}(\alpha - r)\pi^*\right)p,\ p(T) = 1, \\
a' &= -(r + \pi^*(\alpha - r))a,\ a(T) = 1, \\
c' &= -\left((r + \pi^*(\alpha - r))2 + \sigma^2(\pi^*)^2\right)c, c(T) = 1.
\end{aligned}$$

Surprisingly, the solution is $\pi^* = 0$ via $c = a^2$. Note that for this solution, actually U is infinite. However, since $\pi^* U$ is finite, the solution is valid. For this solution,

$$\begin{aligned}
p' &= -rp,\ p(T) = 1, \\
a' &= -ra,\ a(T) = 1, \\
c' &= -2rc,\ c(T) = 1,
\end{aligned}$$

such that

$$\begin{aligned}
p = a &= e^{r(T-t)}, \\
c &= e^{2r(T-t)}.
\end{aligned}$$

This can also be seen from deriving a differential equation for π which gives a DE in the form

$$\begin{aligned}
(\pi^*)' &= k_1(t)\pi^* + k_2(\pi^*)^2 + k_3(\pi^*)^3 \\
\pi^*(T) &= 0,
\end{aligned}$$

with solution

$$\pi^* = 0.$$

This of course makes the case less interesting, although even this is an important insight.

The intuition behind this result is as follows: When the magnitude of the deviations from the mean is smaller than unity, standard deviation punishes these deviations more than does variance. Over an infinitesimal time interval, dt, standard deviation is of order $\sqrt{dt}$, which means that the punishment is so hard that any risk taking is unattractive.

3.3 Endogenous Habit Formation

In this subsection we consider the optimization problem

$$\inf_{\pi}\left(E_{t,x}\left[\frac{1}{2}(X^\pi(T) - xh(t))^2\right]\right).$$

This setup is relevant when investors have a time dependent return target, h. To our knowledge the result is new.

The problem corresponds to the function f given by

$$\begin{aligned}
f(t,x,y,z) &= -\frac{1}{2}z - \frac{1}{2}x^2h^2 + xhy \qquad (24) \\
f_t &= -x^2hh' + xyh', \\
f_x &= -xh^2 + hy, f_{xx} = -h^2, \\
f_y &= xh, f_{xy} = h, \\
f_{yz} &= f_{yy} = f_{xz} = f_{zz} = 0.
\end{aligned}$$

From (10) we can now derive

$$U = 2hG_x - h^2.$$

Plugging U into (11) and (12) we get the following optimal investment candidate and PDE that we need to solve together with (16) and (17),

$$\pi^* = -\frac{\alpha - r}{\sigma^2 x} \frac{F_x + xh^2 - hG}{F_{xx} + h^2 - 2hG_x}, \tag{25}$$

$$F_t = -rx\left(F_x + h^2 x - hG\right) + \frac{1}{2}\theta^2 \frac{\left(F_x + h^2 x - hG\right)^2}{F_{xx} + h^2 - 2hG_x} - x^2 hh' + xh'G,$$

$$F(T,x) = -\frac{1}{2}x^2\left(1 - h(T)\right)^2.$$

We now search for a solution in the form

$$F(t,x) = \frac{1}{2}p(t)x^2, G(t,x) = a(t)x, H(t,x) = c(t)x^2,$$

with $p < 2ah - h^2$, and $a(T) = c(T) = 1$. We know immediately from (24) that the following relation must hold

$$p(t) = 2h(t)a(t) - c(t) - h^2(t), \tag{26}$$

for $c > 0$. The partial derivatives are

$$F_t = \frac{1}{2}p'(t)x^2, F_x = p(t)x, F_{xx} = p(t),$$
$$G_t = a'(t)x, G_x = a(t), G_{xx} = 0,$$
$$H_t = c'(t)x^2, H_x = 2c(t)x, H_{xx} = 2c(t),$$

such that the function U and optimal investment candidate (25) becomes, using (17),

$$U(t,x) = 2h(t)a(t) - h^2(t),$$
$$\pi^*(t) = -\frac{\alpha - r}{\sigma^2}\frac{p(t) - h^2(t) + h(t)a(t)}{p(t) - h^2(t) + 2h(t)a(t)}$$
$$= \frac{\alpha - r}{\sigma^2}\frac{h(t)a(t) - c(t)}{c(t)},$$

where, in the last equation we use (26).

Plugging this strategy and the partial derivatives into (15), (16), and (17) gives the system

$$\frac{1}{2}p' = -\left(r + \pi^*(\alpha - r)/2\right)(ah - c) + h'(a - h),$$
$$a' = -\left(r + \pi^*(\alpha - r)\right)a,$$
$$c' = -\left(\left(r + \pi^*(\alpha - r)\right)2 + \sigma^2(\pi^*)^2\right)c.$$

We can derive the following ODE for π. This is important because then we do not have to calculate a and c in order to derive π^*.

$$\pi^{*\prime} = \frac{\alpha - r}{\sigma^2}\frac{c\left(h'a + ha' - c'\right) - c'(ha - c)}{c^2}$$
$$= \frac{\alpha - r}{\sigma^2}\frac{h'a + ha' - \frac{c'}{c}ha}{c}$$
$$= \frac{\alpha - r}{\sigma^2}\left(\frac{h'}{h} + \left(r + \pi^*(\alpha - r)\right) + \sigma^2(\pi^*)^2\right)\frac{ha}{c}$$
$$= \frac{\alpha - r}{\sigma^2}\left(\frac{h'}{h} + \left(r + \pi^*(\alpha - r)\right) + \sigma^2(\pi^*)^2\right)\left(\pi^*(t)\frac{\sigma^2}{\alpha - r} + 1\right)$$
$$= \left(\frac{h'}{h} + r + \pi^*(\alpha - r) + \sigma^2(\pi^*)^2\right)\left(\pi^*(t) + \frac{\alpha - r}{\sigma^2}\right)$$
$$= k_0(t) + k_1(t)\pi^*(t) + k_2\pi^*(t)^2 + k_3\pi^*(t)^3,$$

with $k_0(t) = (h'/h + r)(\alpha - r)/\sigma^2$, $k_1(t) = h'/h + r + \theta^2$, $k_2 = 2(\alpha - r)$, and $k_3 = \sigma^2$. The boundary condition is $\pi^*(T) = \frac{\alpha - r}{\sigma^2}\left(h(T) - 1\right)$.

Because of the terms $(h'/h + r)\,(\alpha - r)\,/\sigma^2$ the solution is not zero, although $\pi^*\,(T) = 0$ for $h\,(T) = 1$, which is the more meaningful value for $h(T)$. The quantity $-h'/h$ represents the the target rate of return of the investor. Therefore it is reasonable to let $-h'/h$ be a constant larger than r. If $-h'/h = r$, then the optimal strategy *is* zero, precisely because this target can be obtained via a full allocation to the bond.

An example of the optimal strategy can be seen in Figure 1. For comparison, the optimal control in the corresponding pre-commitment case (formalized by (27) below with $h = x_0 e^{0.04T}$) is *initially* $\pi^*(0, x_0) = \left(e^{(0.04 - r)T} - 1\right)(\alpha - r)\,/\sigma^2 \approx 22\%$, but (otherwise) path-dependent. In the pre-commitment case (next section) the optimal allocation, in contrast, tends to zero if performance is good, and vice versa.

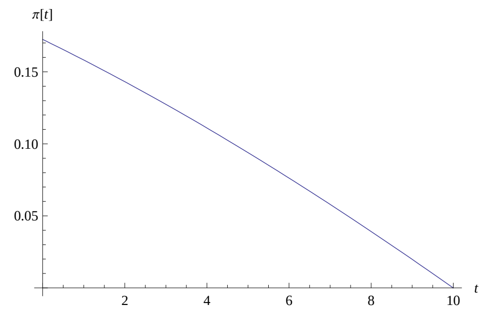

Figure 1: $t \to \pi^*(t)$ for the market $(r, \alpha, \sigma) = (0.02, 0.06, 0.2)$ for an investor with $-h'/h = 4\%$ and horizon $T = 10$.

3.4 Mean-Variance Optimization with pre-commitment

In this subsection we consider the optimization problem formalized by

$$V\,(t, x) = \sup_{\pi} E_{t,x}\left[-\frac{1}{2}\left(X^{\pi}\,(T) - h\right)^2\right] \tag{27}$$

for a constant h.

We start out by explaining how this problem is the 'first step' in solving a variation of mean-variance utility optimization, namely 'with pre-commitment'. Consider the problem

$$V\,(0, x_0) = \sup_{\pi}\left(E\left[X^{\pi}\,(T)\right] - \frac{\gamma}{2} Var\left[X^{\pi}\,(T)\right]\right). \tag{28}$$

The term pre-commitment refers to the target given implicitly by considering the variance as the quadratic deviation from the target $E\left[X^{\pi}\,(T)\right]$. One possibility is to actually update this target with (t, x) on the construction of $V\,(t, x)$. In Subsection 3.1 we updated the target to $E_{t,x}\left[X^{\pi}\,(T)\right]$ in order to formalize the mean-variance optimization problem *without* pre-commitment. An alternative is to refrain from updating the target at all. Therefore we say that we pre-commit ourselves to the target $E_{0,x_0}\left[X^{\pi}\,(T)\right]$ determined at time 0, and we speak of the problem 'with pre-commitment'. This is what we study in this subsection.

First we write the value function of the problem (28) with pre-commitment, i.e. without updating the target

$$V\,(t, x) = \sup_{\pi} E_{t,x}\left[X^{\pi}\,(T) - \frac{\gamma}{2}\left(X^{\pi}\,(T) - E_{0,x_0}\left[X^{\pi}\,(T)\right]\right)^2\right]. \tag{29}$$

This can be rewritten as

$$\begin{aligned} V(t,x) &= \sup_{\pi,K:E_{0,x_0}\left[X^{\pi(K)}(T)\right]=K} E_{t,x}\left[X^{\pi}(T) - \frac{\gamma}{2}\left(X^{\pi}(T)-K\right)^2\right] \\ &= \sup_{\pi,K:E_{0,x_0}\left[X^{\pi(K)}(T)\right]=K} E_{t,x}\left[-\frac{\gamma}{2}X^{\pi}(T)^2 + (1+\gamma K)X^{\pi}(T) - \frac{\gamma}{2}K^2\right]. \end{aligned} \tag{30}$$

The optimization over π and K can be decomposed in two steps: One solves the optimization problem for a general K and finds the optimal strategy $\pi^*(K)$. Then one calculates $E_{0,x_0}\left[X^{\pi^*(K)}(T)\right]$ and determines the optimal K^* as the solution to the nonlinear equation $E_{0,x_0}\left[X^{\pi^*(K^*)}(T)\right] = K^*$. The solution (π^*, K^*) solves the problem formalized by (29).

Rewriting

$$\begin{aligned} V(t,x) &= \gamma \sup_{\pi,K:E_{0,x_0}\left[X^{\pi(K)}(T)\right]=K} E_{t,x}\left[-\frac{1}{2}\left(X^{\pi}(T) - \left(\frac{1}{\gamma}+K\right)\right)^2\right] + \frac{1}{2\gamma} + K \\ &= \gamma \sup_{\pi,h:E_{0,x_0}\left[X^{\pi(h)}(T)\right]=h-\frac{1}{\gamma}} E_{t,x}\left[-\frac{1}{2}\left(X^{\pi}(T)-h\right)^2\right] + h - \frac{1}{2\gamma}, \end{aligned} \tag{31}$$

gives us that solving (27) is the first step of solving (29). The second step is to solve

$$E_{0,x_0}\left[X^{\pi(h)}(T)\right] = h - \frac{1}{\gamma} \tag{32}$$

for h and plug the solution h^* back into π^*. The problem (27) corresponds to the function f given by

$$\begin{aligned} f(t,x,y,z) &= -\left(\frac{1}{2}z + \frac{1}{2}h^2 - hy\right), \\ f_y &= h, f_z = -\frac{1}{2}, \\ f_t &= f_x = f_{xx} = f_{yy} = f_{zz} = f_{xy} = f_{xz} = f_{yz} = 0. \end{aligned}$$

Since all the double derivatives of f are zero we get from (10) that $U = 0$. Plugging $f_t = f_x = U = 0$ into (11) and (12) we get the following optimal investment candidate (and corresponding PDE that we need to solve),

$$\pi^* = -\frac{\alpha - r}{\sigma^2 x}\frac{F_x}{F_{xx}}, \tag{33}$$

$$F_t = -rxF_x + \frac{1}{2}\theta^2\frac{F_x^2}{F_{xx}}, \tag{34}$$

$$F(T,x) = -\frac{1}{2}(x-h)^2.$$

In this particular case the PDE for F does not involve G and H and therefore we do not need to pay attention to the PDEs for G and H. After having derived the solution to (34), this is plugged into (33) to form the optimal investment strategy as a function of (t,x). Plugging this strategy into (16) and (17) results in PDEs characterizing G and H. However, we do not need these characterizations in order to find the optimal investment and the value function.

We now search for a solution in the form

$$F(t,x) = \frac{1}{2}p(t)(x-q(t))^2$$

The partial derivatives are

$$F_t = \frac{1}{2}p'(t)(x-q(t))^2 - q'(t)p(t)(x-q(t)), F_x = p(t)(x-q(t)), F_{xx} = p(t),$$

such that the optimal investment candidate (33) becomes

$$\pi^* x = \frac{\alpha - r}{\sigma^2} \left(q\left(t\right) - x \right). \tag{35}$$

Plugging the partial derivatives into (34) gives

$$\begin{aligned}
&\frac{1}{2} p'\left(t\right)\left(x - q\left(t\right)\right)^2 - q'\left(t\right) p\left(t\right)\left(x - q\left(t\right)\right) \\
&= -p\left(t\right)\left(x - q\left(t\right)\right)^2 r - q\left(t\right) p\left(t\right)\left(x - q\left(t\right)\right) r + \frac{1}{2}\theta^2 p\left(t\right)\left(x - q\left(t\right)\right)^2.
\end{aligned}$$

Collecting terms with $\left(x - q\left(t\right)\right)^2$ and $\left(x - q\left(t\right)\right)$ gives

$$\begin{aligned}
&p'\left(t\right) = \left(-2r + \theta^2\right) p\left(t\right), p\left(T\right) = -1, \\
&q'\left(t\right) = r q\left(t\right), q\left(T\right) = h.
\end{aligned}$$

This system has the solutions

$$\begin{aligned}
p\left(t\right) &= -e^{\left(2r - \theta^2\right)\left(T - t\right)}, \\
q\left(t\right) &= e^{-r\left(T - t\right)} h.
\end{aligned}$$

The full solution can be found by plugging the control (35) into (16) and (17) and guessing a linear and quadratic solution in x to G and H. One finds that

$$\begin{aligned}
G(t,x) &= h\left(1 - e^{-\theta^2\left(T-t\right)}\right) + x e^{\left(r - \theta^2\right)\left(T-t\right)}, \\
H(t,x) &= h^2\left(1 - e^{-\theta^2\left(T-t\right)}\right) + x^2 e^{\left(2r - \theta^2\right)\left(T-t\right)}.
\end{aligned}$$

This is the full solution to the problem without any further specification of h. If we want to solve the mean variance optimization problem with pre-commitment, what remains is to determine h in accordance with (32),

$$\begin{aligned}
G\left(0, x_0\right) = h - \frac{1}{\gamma} &\Leftrightarrow \\
h = \frac{1}{\gamma} e^{\theta^2 T} + x_0 e^{rT} &\Rightarrow \\
q\left(t\right) = e^{rt}\left(x_0 + \frac{1}{\gamma} e^{\left(\theta^2 - r\right)T}\right)&.
\end{aligned}$$

With this representation of q we can now express the optimal wealth and the optimal strategy in terms of the diffusion W. First we note that $q - X^\pi$ follows a geometric Brownian motion,

$$d\left(q\left(t\right) - X^{\pi^*}\left(t\right)\right) = \left(r - \theta^2\right)\left(q\left(t\right) - X^{\pi^*}\left(t\right)\right) dt - \theta\left(q\left(t\right) - X^{\pi^*}\left(t\right)\right) dW\left(t\right).$$

The solution is

$$\begin{aligned}
q\left(t\right) - X^{\pi^*}\left(t\right) &= \left(q\left(0\right) - x_0\right) e^{\left(r - \theta^2 - \frac{1}{2}\theta^2\right)t - \theta W\left(t\right)} \\
&= \frac{1}{\gamma} e^{\left(\theta^2 - r\right)\left(T-t\right)} e^{-\frac{1}{2}\theta^2 t - \theta W\left(t\right)},
\end{aligned}$$

such that

$$\begin{aligned}
X^{\pi^*}\left(t\right) &= q\left(t\right) - \frac{1}{\gamma} e^{\left(\theta^2 - r\right)\left(T-t\right)} e^{-\frac{1}{2}\theta^2 t - \theta W\left(t\right)} \\
&= x_0 e^{rt} + \frac{1}{\gamma}\left(e^{\theta^2 T} e^{-r\left(T-t\right)} - e^{\left(\theta^2 - r\right)\left(T-t\right)} e^{-\frac{1}{2}\theta^2 t - \theta W\left(t\right)}\right).
\end{aligned}$$

Specifically,

$$X^{\pi^*}(T) = x_0 e^{rT} + \frac{1}{\gamma}\left(e^{\theta^2 T} - e^{-\frac{1}{2}\theta^2 T - \theta W(T)}\right). \tag{36}$$

In continuation,

$$\begin{aligned}\pi^* X^{\pi^*}(t) &= \frac{\alpha - r}{\sigma^2}\left(q(t) - X^{\pi^*}(t)\right) \\ &= \frac{\alpha - r}{\sigma^2}\left(\frac{1}{\gamma} e^{\left(\theta^2 - r\right)(T-t)} e^{-\frac{1}{2}\theta^2 t - \theta W(t)}\right).\end{aligned} \tag{37}$$

Recognizing the exponential terms containing the Brownian motion in (36) and (37) as the state price density process times e^{rt}, we can of course express the optimal terminal wealth and the optimal strategy in terms of this process instead. Then the solution in (37) is recognized as the classical solution, see e.g. Basak and Chabakauri (2009), their formulas (37) and (38). The state price representation comes out directly when using the martingale method. In our solution the optimal wealth process can be rewritten only after recognizing the connection between these processes.

The presence of the Brownian motion, or equivalently x_0, in $\pi^* X^{\pi^*}(t)$ shows that the solution is time-inconsistent.

It is easily verified from (35) that both the optimal proportion and the optimal amount invested in stocks is decreasing in wealth. This is a well-known feature and one of the main arguments for not being convinced about the objective concerning practical applications. Actually, this problematic feature is one of the reasons for hunting for alternatives like we did in the preceding subsections.

We conclude by a remark on the mean-variance optimization problem formalized by

$$\inf_{\pi: E[X^\pi(T)] \geq K} Var\left[X^\pi(T)\right].$$

We argue that this problem is equivalent to the problem studied above. By rewriting the problem in terms of a Lagrange multiplier,

$$\begin{aligned} V(0, x_0) &= \inf_{\pi, \lambda: E[X^\pi(T)] = K} E\left[\left(X^\pi(T) - E\left[X^\pi(T)\right]\right)^2 - \lambda X^\pi(T)\right] \\ &= \inf_{\pi, \lambda: E[X^\pi(T)] = K} E\left[\left(X^\pi(T) - K\right)^2 - \lambda X^\pi(T)\right] \\ &= \inf_{\pi, \lambda: E[X^\pi(T)] = K} E\left[X^\pi(T)^2 - (2K + \lambda) X^\pi(T) + K^2\right]. \end{aligned}$$

Now we form the value function with pre-commitment,

$$V(t, x) = \inf_{\pi, \lambda: E[X^\pi(T)] = K} E_{t,x}\left[X^\pi(T)^2 - (2K + \lambda) X^\pi(T) + K^2\right],$$

where the term pre-commitment refers to the fact that the target K equals $E\left[X^\pi(T)\right]$ rather than $E_{t,x}\left[X^\pi(T)\right]$. This problem is essentially equivalent to the problem formalized by

$$V(t, x) = \sup_{\pi, \lambda: E[X^\pi(T)] = K} E_{t,x}\left[-\frac{1}{2}\left(X^\pi(T) - (K + \lambda/2)\right)^2\right]. \tag{38}$$

But this problem is equivalent to the problem (31). In (31) the parameter γ is fixed and the target K is subject to the constraint. In (38) the target K is fixed and the parameter λ is subject to the constraint. So, the optimal portfolios arising from different values of γ in (31) correspond to the optimal portfolios arising from different values of K in (38).

4 Collective Objectives

In this section we apply Theorem 1 to a new set of problems that arise for a collective of heterogenous investors. We think of a group of n investors who, despite their different attitudes towards risk, invest in the same mutual fund. The task is to form an optimal investment strategy for this mutual fund. Such a study is e.g. relevant for compulsory pension schemes.

For simplicity we assume that all investors bring in the same amount x_0/n, and that they all participate in the fund over the same period. Also, they share the same beliefs about the financial market. At the end of the optimization horizon the terminal wealth $X^{\pi}(T)$ is, correspondingly, distributed equally to all investors, such that each investor receives $X^{\pi}(T)/n$. Thus, the risk *sharing* is fixed and is not subject to optimization. However, the aggregate wealth $X^{\pi}(T)$ is not fixed and is subject to optimization via the investment strategy π. It is important to understand that we are considering the problem of optimal investment for a group of investors which is marginal to the total number of investors in the economy. Therefore, there is no equilibrium theory or asset price formation taking place here. Equilibrium asset prices are given and this marginal group of investors with heterogenous risk aversions plays the investment game together for one reason or another (e.g. in order to save on transaction costs (widely defined) or because they are forced to).

The question is now, what is the objective of the group. A first naive idea is to add up the indirect utility from each investor to achieve the value function.

$$\sup_{\pi}\sum_{i=1}^{n}E_{t,x}\left[u_i\left(X^{\pi}\left(T\right)/n\right)\right]=\sup_{\pi}E_{t,x}\left[\sum_{i=1}^{n}u_i\left(X^{\pi}\left(T\right)/n\right)\right]. \tag{39}$$

This problem can in principle be solved via standard techniques, but it suffers from serious drawbacks: There is no economic point in adding up different utility functions. For each investor, the utility function expresses his preferences, but it is merely ordinal. Thus, since the utility functions are not comparable, they tell nothing about preferences across the group of investors. A simple check of economic reasonability is the unit of the terms in the sum. For heterogenous investors we are adding up different functions of the currency unit, and this is also a warning that the formulation (39) is completely useless.

The idea that we will introduce here is to align each investor's indirect utility before summation by calculating his certainty equivalent. Thus, we propose instead the formalization

$$\sup_{\pi}\sum_{i=1}^{n}u_i^{-1}\left(E_{t,x}\left[u_i\left(X^{\pi}\left(T\right)/n\right)\right]\right). \tag{40}$$

This makes economic sense: At time t we are adding up certain time t-amounts which are definitely comparable. From a mathematical point of view, though, the problem (40) seems more challenging, due to the non-linearity of the utility functions, but our Theorem 1 is able to cope with that.

We re-emphasize that the proportional division of terminal wealth is pre-imposed, so it is not possible to increase group utility by assigning all wealth to the more risk-tolerant agent. There may exist more optimal risk sharing rules - especially should one know more about the agents' endowments. Still, the simple rule that we have outlined is highly relevant from a practical perspective.

4.1 A Collective of Exponential Utility Investors

For exponential utility with coefficients of absolute risk aversion $\xi_i > 0$ and $n = 2$ the problem (40) is

$$\sup_{\pi}\left(\frac{-1}{\xi_1}\log E\left[e^{-\xi_1 X^{\pi}(T)/2}\right]+\frac{-1}{\xi_2}\log E\left[e^{-\xi_2 X^{\pi}(T)/2}\right]\right).$$

This corresponds to the function f given by

$$\begin{aligned}
f\left(y,z\right)&=-\frac{\log y}{\xi_1}-\frac{\log z}{\xi_2},\\
f_t&=f_x=f_{xx}=f_{xy}=f_{xz}=f_{yz}=0,\\
f_y&=-\frac{1}{\xi_1 y},f_{yy}=\frac{1}{\xi_1 y^2},\\
f_z&=-\frac{1}{\xi_2 z},f_{zz}=\frac{1}{\xi_2 z^2}.
\end{aligned}$$

From (10) we can now derive

$$U = \frac{1}{\xi_1}\left(\frac{G_x}{G}\right)^2 + \frac{1}{\xi_2}\left(\frac{H_x}{H}\right)^2.$$

Plugging $f_t = f_x = 0$ and U into (11) we get the following optimal candidate,

$$\pi^* = -\frac{\alpha - r}{\sigma^2 x}\frac{F_x}{F_{xx} - \frac{1}{\xi_1}\left(\frac{G_x}{G}\right)^2 - \frac{1}{\xi_2}\left(\frac{H_x}{H}\right)^2}. \tag{41}$$

With this specification of π^* we now search for a solution to (15), (16), and (17) in the form

$$F(t,x) = p(t)x + q(t), G(t,x) = e^{g_1(t)x + g_2(t)}, H(t,x) = e^{h_1(t)x + h_2(t)}.$$

The partial derivatives are

$$\begin{aligned}
F_t &= p'(t)x + q'(t), F_x = p(t), F_{xx} = 0,\\
G_t &= e^{g_1(t)x+g_2(t)}\left(g_1'(t)x + g_2'(t)\right),\\
G_x &= e^{g_1(t)x+g_2(t)} g_1(t), G_{xx} = e^{g_1(t)x+g_2(t)} g_1^2(t),\\
H_t &= e^{h_1(t)x+h_2(t)}\left(h_1'(t)x + h_2'(t)\right),\\
H_x &= e^{h_1(t)x+h_2(t)} h_1(t), H_{xx} = e^{h_1(t)x+h_2(t)} h_1^2(t),
\end{aligned}$$

such that the optimal investment candidate (41) becomes

$$\pi^* x = \frac{\alpha - r}{\sigma^2}\frac{p(t)}{\frac{1}{\xi_1}g_1^2(t) + \frac{1}{\xi_2}h_1^2(t)}. \tag{42}$$

Plugging this strategy into (15), (16), and (17), leads to ordinary differential equations for p, g_1, and h_1, with terminal conditions and solutions

$$\begin{aligned}
p'(t) &= -rp(t); p(T) = 1: \ p(t) = e^{r(T-t)},\\
g_1'(t) &= -rg_1(t); g_1(T) = -\frac{\xi_1}{2}: \ g_1(t) = -\frac{\xi_1}{2}e^{r(T-t)},\\
h_1'(t) &= -rh_1(t); h_1(T) = -\frac{\xi_2}{2}: \ h_1(t) = -\frac{\xi_2}{2}e^{r(T-t)}.
\end{aligned}$$

Plugging these into (42) yields

$$\pi^* x = 2\frac{\alpha - r}{\sigma^2}\frac{e^{-r(T-t)}}{\bar{\xi}},$$

with $\bar{\xi} = (\xi_1 + \xi_2)/2$ defining the average risk aversion.

This strategy may be compared to the classical solution for a single investor with risk aversion ξ who invests optimally the amount $\frac{\alpha-r}{\sigma^2}\frac{e^{-r(T-t)}}{\xi}$. We see that the collective of investors calculates an average absolute risk aversion coefficient $\bar{\xi}$, and then invests two times the amount that such an average investor would, i.e. one (time) for each participant. Notice that this strategy is not the simple average of individually optimal strategies.

The result above can easily be extended to the case of n investors characterized by exponential utility with coefficients $\xi_1, \ldots \xi_n$. They should invest n times the amount resulting from the average risk aversion

$$\pi^* x = n\frac{\alpha - r}{\sigma^2}\frac{e^{-r(T-t)}}{\bar{\xi}},$$

with $\bar{\xi} = \frac{1}{n}\sum_{i=1}^n \xi_i$.

Another interpretation of this expression is that it corresponds to an individual who is initially n times richer than each individual, and whose coefficient of absolute risk aversion is thus n times lower, namely $\bar{\xi}/n$.

For the full solution we also solve the ordinary differential equations for q, g_2, and h_2, and get

$$q(t) = \frac{n\theta^2}{2\bar{\xi}}(T-t),$$
$$g_2(t) = -\frac{\theta^2(T-t)\xi_1}{2\bar{\xi}^2}\left(2\bar{\xi}-\xi_1\right)$$
$$h_2(t) = -\frac{\theta^2(T-t)\xi_2}{2\bar{\xi}^2}\left(2\bar{\xi}-\xi_2\right),$$

with $n=2$.

With n investors the group-optimal discounted certainty equivalent is thus

$$\begin{aligned} e^{-r(T-t)}F(t,x) &= n\left(\frac{x}{n} + e^{-r(T-t)}\frac{\theta^2}{2\bar{\xi}}(T-t)\right) \\ &= e^{-r(T-t)}\sum_{i=1}^{n}\frac{-1}{\xi_i}\left[\frac{-\xi_i}{n}e^{r(T-t)}x - \frac{\theta^2(T-t)\xi_i}{2\bar{\xi}^2}\left(2\bar{\xi}-\xi_i\right)\right] \\ &= \sum_{i=1}^{n}\left[\frac{x}{n} + e^{-r(T-t)}\frac{\theta^2(T-t)}{2\bar{\xi}^2}\left(2\bar{\xi}-\xi_i\right)\right], \end{aligned}$$

with the individual terms in the sum corresponding to the i^{th} individual's certainty equivalent. On the other hand, if each individual invests on his own, he obtains the comparable optimal discounted certainty equivalent

$$\frac{x}{n} + e^{-r(T-t)}\frac{\theta^2}{2\xi_i}(T-t),$$

such that his relative loss (of discounted certainty equivalent after subtraction of x/n) from entering the collective is

$$1 - \frac{e^{-r(T-t)}\xi_i^{-1}\frac{\theta^2(T-t)\xi_i}{2\bar{\xi}^2}\left(2\bar{\xi}-\xi_i\right)}{e^{-r(T-t)}\frac{\theta^2}{2\xi_i}(T-t)} = \left(1-\frac{\xi_i}{\bar{\xi}}\right)^2,$$

which could be compared to the estimated gains from economies of scale. These losses are - unsurprisingly - independent of initial wealth. In the case of two investors they both lose the same proportion, but when there are more agents some can be hit substantially larger than others (and some may not suffer at all, of course).

As in the mean-variance case (Subsection 3.1) it is relevant to let the coefficients of absolute risk aversion depend on t, x. However, our methodology cannot cope with this setting. For a single investor the example is a special case of Björk and Murgoci (2008).

4.2 A Collective of Power Utility Investors

As another, and perhaps more interesting, example consider a collective of power utility investors with coefficients of relative risk aversion $1-\gamma_i \in (0,\infty)\backslash\{1\}$. For illustration we consider a small collective with $n=2$, for which the problem (40) is

$$\sup_{\pi}\frac{1}{2}\left[\left(E_{t,x}\left[\left(X^{\pi}(T)\right)^{\gamma_1}\right]\right)^{\gamma_1^{-1}} + \left(E_{t,x}\left[\left(X^{\pi}(T)\right)^{\gamma_2}\right]\right)^{\gamma_2^{-1}}\right].$$

This corresponds to the function f given by

$$f(y,z) = \frac{1}{2}\left(y^{\gamma_1^{-1}} + z^{\gamma_2^{-1}}\right), \tag{43}$$
$$f_t = f_x = f_{xx} = f_{xy} = f_{xz} = f_{yz} = 0,$$
$$f_y = \frac{1}{2}\gamma_1^{-1}y^{\frac{1-\gamma_1}{\gamma_1}},\ f_{yy} = \frac{1}{2}\frac{1-\gamma_1}{\gamma_1^2}y^{\frac{1-2\gamma_1}{\gamma_1}},$$
$$f_z = \frac{1}{2}\gamma_2^{-1}z^{\frac{1-\gamma_2}{\gamma_2}},\ f_{zz} = \frac{1}{2}\frac{1-\gamma_2}{\gamma_2^2}z^{\frac{1-2\gamma_2}{\gamma_2}}.$$

From (10) we can now derive

$$U=\frac{1}{2}\frac{1-\gamma_1}{\gamma_1^2}G^{\gamma_1^{-1}}\left(\frac{G_x}{G}\right)^2+\frac{1}{2}\frac{1-\gamma_2}{\gamma_2^2}H^{\gamma_2^{-1}}\left(\frac{H_x}{H}\right)^2.$$

Plugging $f_t=f_x=0$ and U into (11) we get the following optimal candidate,

$$\pi^*=-\frac{\alpha-r}{\sigma^2 x}\frac{F_x}{F_{xx}-\left(\frac{1-\gamma_1}{\gamma_1^2}G^{\gamma_1^{-1}}\left(\frac{G_x}{G}\right)^2+\frac{1-\gamma_2}{\gamma_2^2}H^{\gamma_2^{-1}}\left(\frac{H_x}{H}\right)^2\right)}. \tag{44}$$

With this specification of π^* we now search for a solution to (15), (16), and (17) in the form

$$F(t,x)=p(t)\,x, G(t,x)=a^{\gamma_1}(t)\,x^{\gamma_1}, H(t,x)=c^{\gamma_2}(t)\,x^{\gamma_2},$$

The partial derivatives are

$$\begin{aligned}
F_t&=p'(t)\,x, F_x=p(t)\,, F_{xx}=0,\\
G_t&=\gamma_1 a^{\gamma_1-1}(t)\,a'(t)\,x^{\gamma_1},\\
G_x&=\gamma_1 a^{\gamma_1}(t)\,x^{\gamma_1-1}, G_{xx}=\gamma_1(\gamma_1-1)\,a^{\gamma_1}(t)\,x^{\gamma_1-2},\\
H_t&=\gamma_2 c^{\gamma_2-1}(t)\,c'(t)\,x^{\gamma_2},\\
H_x&=\gamma_2 c^{\gamma_2}(t)\,x^{\gamma_2-1}, H_{xx}=\gamma_2(\gamma_2-1)\,c^{\gamma_2}(t)\,x^{\gamma_2-2},
\end{aligned}$$

and $p=(a+c)/2$, such that the optimal candidate (44) becomes

$$\begin{aligned}
\pi^*&=\frac{\alpha-r}{\sigma^2}\frac{a(t)+c(t)}{(1-\gamma_1)\,a(t)+(1-\gamma_2)\,c(t)}\\
&=\frac{\alpha-r}{\sigma^2}\frac{1}{1-\gamma(t)},
\end{aligned}$$

with $\gamma(t)$ defined as a weighted average of the underlying coefficients with time-dependent weights,

$$\gamma(t)=\frac{a(t)\,\gamma_1+c(t)\,\gamma_2}{a(t)+c(t)}.$$

This formulation means that (in contrary to the exponential case) there can never be an agent, who can be taken to be representative for the collective over the entire period.

Plugging this strategy into (15), (16), and (17), leads to a system of ordinary differential equations for p, a, and c, with terminal conditions. The differential equations for a and c can be solved isolated from p and are sufficient for determination of π. We find the following representation in terms of γ,

$$\begin{aligned}
a'(t)&=-\left(r+\frac{\theta^2}{1-\gamma(t)}-\frac{\theta^2}{2}\frac{1-\gamma_1}{(1-\gamma(t))^2}\right)a(t); a(T)=1,\\
c'(t)&=-\left(r+\frac{\theta^2}{1-\gamma(t)}-\frac{\theta^2}{2}\frac{1-\gamma_2}{(1-\gamma(t))^2}\right)c(t); c(T)=1.
\end{aligned}$$

We have no explicit solution to the two-dimensional system of ordinary equations. We can however characterize the solution a bit further by calculating an ODE for the quantity $w=\frac{a}{a+c}$, which is the weight on agent 1's coefficient of relative risk aversion in the formation of the group's ditto:

$$w'=w(1-w)(\gamma_2-\gamma_1)\frac{\theta^2}{2}\left[1-\gamma_2+w(\gamma_2-\gamma_1)\right]^{-2},\ w(T)=1/2,$$

with the property that for $\gamma_1>\gamma_2$, w is a decreasing function of time so that the more risk tolerant agent has the larger weight. For $w>\frac{1-\gamma_2}{1-\gamma_1+1-\gamma_2}$, the weight on agent 1's individually optimal strategy is larger than a half. This is equivalent to local concavity of w which will occur for sufficiently long time horizons.

For n investors n differential equations can be reduced to $n-1$ using this technique, but the advantage is not nearly as obvious.

An illustration of the development over time of the weight can be seen in Figure 2, whereas Table 1 shows the corresponding group strategies and certainty equivalents, and contrasts them to those of the individuals forming the collective. Notice that Figure 2 considers $t = 100$, but this of course includes all shorter horizons as well. The figure reveals that the collective's relative risk aversion is rather slowly changing over time - corresponding to an "almost constant" relative allocation to risky assets.

When participating in the group, the discounted certainty equivalent of individual 1 is $e^{-r(T-t)}a(t)x$, while as an individual he would be indifferent between participating in the lottery and receiving $x \exp\left(\theta^2(T-t)/(2(1-\gamma))\right)$. Depending on the measurement of loss one or the other investor will be worst off.

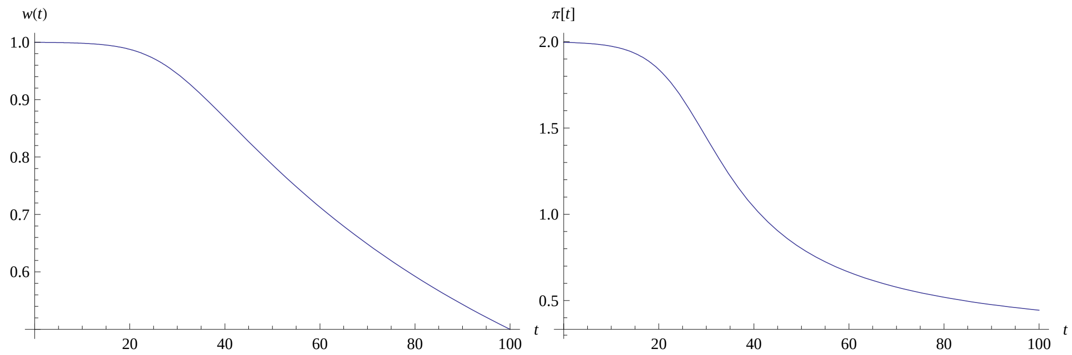

Figure 2: Left panel: $t \to w(t)$. Right panel: $t \to \pi^*(t)$. The market is $(r, \alpha, \sigma) = (0.02, 0.06, 0.2)$, and the collective is formed by two investors with $\gamma_1 = 0.5$, $\gamma_2 = -3$, and horizon $T = 100$. The inflection point is around $t = 41.5$.

	$\pi(0)$	$\pi(T-10)$	$\pi(T)$	CE $(T-10, 1)$ (discounted)
Agent 1	200%	200%	200%	1.49
Agent 2	33%	33%	33%	1.07
Collective	199.6%	61%	57%	2.25 (1.22 resp. 1.03)

Table 1: Optimal allocation to risky assets and corresponding optimal certainty equivalents for agents with relative risk aversions $\gamma_1 = 0.5$, $\gamma_2 = -3$, and horizon $T = 100$. The third row shows the corresponding figures for the group formed by the two agents. The market is $(r, \alpha, \sigma) = (0.02, 0.06, 0.2)$

4.3 A Collective of Mean-Variance Utility Investors without pre-commitment

We paid a lot of attention to the mean-variance utility investor in the previous section. Let us see what happens if we apply our certainty equivalent approach to a group of heterogenous mean-variance utility investors. This becomes particularly simple in the case without pre-commitment, because the certainty equivalence for an investor with expected terminal wealth x is x itself. Thus, the utility inversion just becomes the identity function, and we study the problem

$$\sup_{\pi} \sum_{i=1}^{n} \left(E_{t,x}\left[\frac{X^{\pi}(T)}{n}\right] - \frac{\gamma_i}{2} Var_{t,x}\left[\frac{X^{\pi}(T)}{n}\right]\right)$$

$$= \sup_{\pi} \left(E_{t,x}\left[X^{\pi}(T)\right] - \frac{\overline{\gamma}/n}{2} Var_{t,x}\left[X^{\pi}(T)\right]\right),$$

with $\overline{\gamma} = \frac{1}{n}\sum_{i=1}^{n}\gamma_i$ defining the average risk aversion. But this problem is equivalent to the problem of a single investor with wealth x and risk aversion $\overline{\gamma}/n$. The optimal investment strategy then becomes

$$\pi^* x = \frac{n}{\overline{\gamma}}\frac{\alpha - r}{\sigma^2}e^{-r(T-t)}.$$

This should be compared with the solution for a single investor with risk aversion γ, who invests optimally the amount $\pi^* x = \frac{1}{\gamma}e^{-r(T-t)}(\alpha - r)/\sigma^2$. As was the case for exponential utility collectives, we find that the group-optimal amount invested in stocks is found by using the average risk aversion $\overline{\gamma}$, and then investing this amount for each of the n participants.

Since mean-variance is not a real utility function there need not be a loss associated with joining a group. An individual investor, i, will have an optimal discounted certainty equivalent of $\frac{x}{n} + e^{-r(T-t)}\frac{\theta^2}{\gamma_i}(T-t)$, while as a group member his corresponding "indifference amount" will be $\frac{x}{n} + e^{-r(T-t)}\frac{\theta^2}{\bar{\gamma}}(T-t)$, so that he will incur a loss (again, in certainty-equivalent terms) by joining the group iff $\gamma_i < \bar{\gamma}$, i.e if he is less cautious than the group as a whole.

4.4 A Collective of Mean-Variance Utility Investors with pre-commitment

We can also consider the mean-variance utility with pre-commitment *for a collective.* Here it becomes important in which order we implement the different arguments. Does each investors realize that the utility inversion of a mean-variance utility is the identity function *before* he decides to pre-commit himself to his time 0-target? Or does he pre-commit to his time 0-target for thereafter to realize that the utility inversion is no longer just the identity function?

If we implement the identity utility inversion, we get the problem

$$\begin{aligned}
&\sup_{\pi}\sum_{i=1}^{n}\left(E_{t,x}\left[\frac{X^\pi(T)}{n}\right] - \frac{\gamma_i}{2}E_{t,x}\left[\frac{X^\pi(T)}{n} - E_{0,x_0}\left[\frac{X^\pi(T)}{n}\right]\right]^2\right)\\
&= \sup_{\pi:E_{0,x_0}[X^\pi(T)]=K}\sum_{i=1}^{n}\left(E_{t,x}\left[\frac{X^\pi(T)}{n}\right] - \frac{\gamma_i}{2}E_{t,x}\left[\frac{X^\pi(T)}{n} - \frac{K}{n}\right]^2\right)\\
&= \sup_{\pi:E_{0,x_0}[X^\pi(T)]=K}\left(E_{t,x}\left[X^\pi(T)\right] - \frac{\overline{\gamma}}{2n}E_{t,x}\left[X^\pi(T) - K\right]^2\right)\\
&= \sup_{\pi,h:E_{0,x_0}\left[X^{\pi(h)}(T)\right]=h-\frac{n}{\overline{\gamma}}} E_{t,x}\left[-\frac{1}{2}\left(X^\pi(T) - h\right)^2\right]
\end{aligned}$$

with $\overline{\gamma} = \frac{1}{n}\sum_{i=1}^{n}\gamma_i$ defining the average risk aversion. But this problem is equivalent to the problem of a single investor with wealth x and risk aversion $\overline{\gamma}/n$. The optimal investment strategy then becomes

$$\pi^* x = \frac{\alpha - r}{\sigma^2}\left(q(t) - x\right)$$

with

$$q(t) = e^{rt}\left(x_0 + \frac{n}{\overline{\gamma}}e^{\left(\theta^2 - r\right)T}\right).$$

This can be compared to the solution for a single investor with risk aversion γ_i and initial wealth x_0/n, who invests optimally the amount $\pi^* x/n = \frac{\alpha - r}{\sigma^2}\left(q_i(t) - x/n\right)$ with $q_i(t) = e^{rt}\left(x_0/n + \frac{1}{\gamma_i}e^{\left(\theta^2 - r\right)T}\right)$. We see that the collective of investors calculate an average target process $\overline{q}$ based on the average aversion $\overline{\gamma}$, $\overline{q}(t) = e^{rt}\left(x_0/n + \frac{1}{\overline{\gamma}}e^{\left(\theta^2 - r\right)T}\right)$, and then invests n times this amount, $\pi^* x = n\frac{\alpha - r}{\sigma^2}\left(\overline{q}(t) - x/n\right) = \frac{\alpha - r}{\sigma^2}\left(q(t) - x\right)$.

The alternative is to start with the pre-commitment such that objective of investor i, before starting the investment collective, is

$$V_i(t,x) = \sup_{\pi,h_i:E\left[X^{\pi(h_i)}(T)\right]=h_i - \frac{1}{\gamma_i}} E_{t,x}\left[-\frac{1}{2}\left(X^\pi(T) - h_i\right)^2\right].$$

Now the utility inversion is no longer the identity function, and the case turns out to be surprisingly difficult to deal with. First we have to assume that $\frac{X^{\pi}}{n} \leq h_i$ a.s. for all i. Then the collective of investors faces the problem

$$V(t,x) = \sup_{\pi, h_i : E\left[\frac{X^{\pi(h_i)}(T)}{n}\right] = h_i - \frac{1}{\gamma_i}} \sum_{i=1}^{n} \left(h_i + \sqrt{E_{t,x}\left[\left(\frac{X^{\pi}(T)}{n} - h_i\right)^2\right]} \right),$$

which seems intractable.

5 Concluding remarks

Björk and Murgoci (2008) point out that to any non-standard problem within their set of study corresponds a standard problem. Here we argue that this also hold in our case. Rearranging the terms of (6) yields

$$F_t = \inf_{\pi} \left[-\left(r + \pi(\alpha - r)\right) x F_x - \frac{1}{2}\sigma^2\pi^2 x^2 F_{xx} + f_t + \left(r + \pi(\alpha - r)\right) x f_x + \frac{1}{2}\sigma^2\pi^2 x^2 U \right].$$

One can recognize this as the standard HJB equation to the problem

$$\max_{\pi} E_{t,x} \left[\begin{array}{c} -\int_t^T \left(f_s + \left(r + \pi(\alpha - r)\right) X(s) f_x + \frac{1}{2}\sigma^2\pi^2 X(s)^2 U(f,y,z) \right) ds \\ + f\left(T, X(T), g(X(T)), h(X(T))\right) \end{array} \right], \tag{45}$$

with appropriate arguments. Björk and Murgoci (2008) calculated specifically the equivalent standard problem for the mean-variance case and their result can be recognized in (45). Formalizing the 'extra' terms in Bellman-like equation (6) as 'utility of consumption' is straightforward here, and probably in cases much more involved than ours likewise. However, it is of only marginal interest since we do not know any examples where the standard problem induced by a non-standard problem has a meaningful economic interpretation in its own respect.

In this paper we have concentrated on the pure investment problem. In Björk and Murgoci (2008), consumption is also taken into account. Their preferences over consumption contribute to the inconsistency only via dependence on wealth (like endogenous habit formation). More generally, inconsistency could also arise from taking a non-linear function of the expected utility of consumption. This is a natural subject for further research.

We have here completely accepted the non-standard problem as meaningful. The game theoretical foundations for interpretation of the non-standard problem is taken as given in our presentation and we refer to Björk and Murgoci (2008) for theoretical considerations in this regard. Once having accepted the problem as meaningful we are allowed to attack it directly in continuous time with more or less standard control theoretical techniques. Therefore the generalized HJB equation and the examples of its solution stand out as the primary contribution of our paper.

References

[1] S. Basak and G. Chabakauri. Dynamic Mean-Variance Asset Allocation. Technical report, EFA 2007 Ljubljana Meetings; AFA 2009 San Fransisco Meetings paper. Available at SSRN: http://ssrn.com/abstract=965926, 2009.

[2] T. Björk and A. Murgoci. A General Theory of Markovian Time Inconsistent Stochastic Control Problems. Technical report, Group Essec. Available at http://econtent.essec.fr/mediabanks/ESSEC-PDF/Enseignement%20et%20Recherche/Enseignement/Departement/seminaire/Finance/2008-2009/Tomas_Bjork-Seminaire.pdf, 2008.

[3] T. Björk, A. Murgoci, and X. Zhou. Time Inconsistent Control and Mean Variance Portfolios with State Dependent Risk Aversion. Technical report, KTH. Available at: `http://www.math.kth.se/pdefinance/2009/presentations/Bjork.pdf`, 2009.

[4] R. A. Pollak. Consistent Planning. *The Review of Economic Studies*, 35(2):201–208, 1968.

[5] J. M. Solano and J. Navas. Consumption and portfolio rules for time-inconsistent investors. *European Journal of Operational Research*, 201:860–872, 2010.

[6] R. H. Strotz. Myopia and Inconsistency in Dynamic Utility Maximization. *The Review of Economic Studies*, 23(3):165–180, 1956.

Appendix

Proof of Theorem 1. Consider an arbitrary admissible strategy π.

1. First we argue that if there exists a function $Y^{\pi}(t,x)$ such that

$$Y_t^{\pi} = -\left(r+\pi\left(\alpha-r\right)\right)xY_x^{\pi} - \frac{1}{2}\sigma^2\pi^2x^2Y_{xx}^{\pi}, \tag{46}$$

$$Y^{\pi}(T,x) = g(x), \tag{47}$$

then

$$Y^{\pi}(t,x) = y^{\pi}(t,x). \tag{48}$$

Namely,

$$\begin{aligned} Y^{\pi}(t,X(t)) &= -\int_t^T dY^{\pi}(s,X^{\pi}(s)) + Y^{\pi}(T,X^{\pi}(T)) \\ &= -\int_t^T \begin{pmatrix} Y_s^{\pi}(s,X^{\pi}(s))\,ds \\ +Y_x^{\pi}(s,X^{\pi}(s))\begin{pmatrix} (r+\pi(s)(a-r))X^{\pi}(s)\,ds \\ +\pi(s)\sigma X^{\pi}(s)\,dW(s)\end{pmatrix} \\ +\frac{1}{2}Y_{xx}^{\pi}(s,X^{\pi}(s))\sigma^2\pi^2(s)X^{\pi}(s)^2\,ds \end{pmatrix} \\ &+ Y^{\pi}(T,X^{\pi}(T)). \end{aligned}$$

Inserting (46) and (47) gives

$$Y^{\pi}(t,X(t)) = -\int_t^T \pi(s)\sigma X^{\pi}(s)\,dW(s) + g(X^{\pi}(T)). \tag{49}$$

Now, taking conditional expectation on both sides gives

$$Y^{\pi}(t,x) = E_{t,x}\left[g\left(X^{\pi}(T)\right)\right] = y^{\pi}(t,x).$$

From similar arguments (replace y and Y by z and Z) we get that if there exists a function $Z^{\pi}(t,x)$ such that

$$Z_t^{\pi} = -\left(r+\pi\left(\alpha-r\right)\right)xZ_x^{\pi} - \frac{1}{2}\sigma^2\pi^2x^2Z_{xx}^{\pi}, \tag{50}$$

$$Z^{\pi}(T,x) = h(x), \tag{51}$$

then

$$Z^{\pi}(t,x) = z^{\pi}(t,x). \tag{52}$$

2. Second we obtain an expression for

$$f\left(t,X^{\pi}\left(t\right),y^{\pi}\left(t,X^{\pi}\left(t\right)\right),z^{\pi}\left(t,X^{\pi}\left(t\right)\right)\right).$$

From (48) and (52) we have that this equals

$$f\left(t,X^{\pi}\left(t\right),Y^{\pi}\left(t,X^{\pi}\left(t\right)\right),Z^{\pi}\left(t,X^{\pi}\left(t\right)\right)\right).$$

Since f is sufficiently differentiable, then by Ito

$$\begin{aligned}
&f\left(t,X^{\pi}\left(t\right),y^{\pi}\left(t,X^{\pi}\left(t\right)\right),z^{\pi}\left(t,X^{\pi}\left(t\right)\right)\right)\\
&=-\int_{t}^{T}df\left(s,X^{\pi}\left(s\right),Y^{\pi}\left(s,X^{\pi}\left(s\right)\right),Z^{\pi}\left(s,X^{\pi}\left(s\right)\right)\right)\\
&+f\left(T,X^{\pi}\left(T\right),Y^{\pi}\left(T,X^{\pi}\left(T\right)\right),Z^{\pi}\left(T,X^{\pi}\left(T\right)\right)\right)\\
&=-\int_{t}^{T}\left(\begin{array}{c}\left(f_{s}+f_{y}Y_{s}^{\pi}+f_{z}Z_{s}^{\pi}\right)ds\\+\left(f_{x}+f_{y}Y_{x}^{\pi}+f_{z}Z_{x}^{\pi}\right)dX^{\pi}\left(s\right)\\+\frac{1}{2}\left(\begin{array}{c}f_{xx}+2f_{xy}Y_{x}^{\pi}+2f_{xz}Z_{x}^{\pi}\\+f_{y}Y_{xx}^{\pi}+f_{z}Z_{xx}^{\pi}+f_{yy}\left(Y_{x}^{\pi}\right)^{2}\\+2f_{yz}Y_{x}^{\pi}Z_{x}^{\pi}+f_{zz}\left(Z_{x}^{\pi}\right)^{2}\end{array}\right)\sigma^{2}\pi^{2}\left(s\right)X^{\pi}\left(s\right)^{2}ds\end{array}\right)\\
&+f\left(T,X^{\pi}\left(T\right),Y^{\pi}\left(T,X^{\pi}\left(T\right)\right),Z^{\pi}\left(T,X^{\pi}\left(T\right)\right)\right).
\end{aligned}$$

where we have skipped some arguments under the integral. Inserting (46), (47), (50), (51) and (2) we have that

$$\begin{aligned}
&f\left(t,X^{\pi}\left(t\right),y^{\pi}\left(t,X^{\pi}\left(t\right)\right),z^{\pi}\left(t,X^{\pi}\left(t\right)\right)\right)\\
&=-\int_{t}^{T}\left(\begin{array}{c}\left(\begin{array}{c}f_{s}+f_{y}\left(-\left(r+\pi(s)\left(\alpha-r\right)\right)xY_{x}^{\pi}-\frac{1}{2}\sigma^{2}\pi(s)^{2}x^{2}Y_{xx}^{\pi}\right)\\+f_{z}\left(-\left(r+\pi(s)\left(\alpha-r\right)\right)xZ_{x}^{\pi}-\frac{1}{2}\sigma^{2}\pi(s)^{2}x^{2}Z_{xx}^{\pi}\right)\end{array}\right)ds\\+\left(f_{x}+f_{y}Y_{x}^{\pi}+f_{z}Z_{x}^{\pi}\right)\left(\begin{array}{c}\left(r+\pi\left(s\right)\left(\alpha-r\right)\right)X^{\pi}\left(s\right)dt\\+\pi\left(s\right)\sigma X^{\pi}\left(s\right)dW\left(s\right)\end{array}\right)\\+\frac{1}{2}\left(\begin{array}{c}f_{xx}+2f_{xy}Y_{x}^{\pi}+2f_{xz}Z_{x}^{\pi}\\+f_{y}Y_{xx}^{\pi}+f_{z}Z_{xx}^{\pi}+f_{yy}\left(Y_{x}^{\pi}\right)^{2}\\+2f_{yz}Y_{x}^{\pi}Z_{x}^{\pi}+f_{zz}\left(Z_{x}^{\pi}\right)^{2}\end{array}\right)\sigma^{2}\pi^{2}\left(s\right)X^{\pi}\left(s\right)ds\end{array}\right)\\
&+f\left(T,X^{\pi}\left(T\right),g\left(X^{\pi}\left(T\right)\right),h\left(X^{\pi}\left(T\right)\right)\right).
\end{aligned}$$

Abbreviating and inserting (10) we get

$$\begin{aligned}
&f\left(t,X^{\pi}\left(t\right),y^{\pi}\left(t,X^{\pi}\left(t\right)\right),z^{\pi}\left(t,X^{\pi}\left(t\right)\right)\right)\\
&=-\int_{t}^{T}\left(\begin{array}{c}f_{s}ds+f_{x}\left(r+\pi\left(s\right)\left(\alpha-r\right)\right)X^{\pi}\left(s\right)ds\\+\left(f_{x}+f_{y}Y_{x}^{\pi}+f_{z}Z_{x}^{\pi}\right)\pi\left(s\right)\sigma X^{\pi}\left(s\right)dW\left(s\right)\\+\frac{1}{2}U\left(f,Y^{\pi},Z^{\pi}\right)\sigma^{2}\pi^{2}\left(s\right)X^{\pi}\left(s\right)rds\end{array}\right)\\
&+f\left(T,X^{\pi}\left(T\right),g\left(X^{\pi}\left(T\right)\right),h\left(X^{\pi}\left(T\right)\right)\right).
\end{aligned}\tag{53}$$

3. Third, we establish on the basis of (53) that

$$F\left(t,x\right)\geq\sup_{\pi}f\left(t,x,y^{\pi}\left(t,x\right),z^{\pi}\left(t,x\right)\right).$$

An Ito calculation on F gives that

$$\begin{aligned}
F\left(t,X^{\pi}\left(t\right)\right)&=-\int_{t}^{T}dF\left(s,X^{\pi}\left(s\right)\right)+F\left(T,X^{\pi}\left(T\right)\right)\\
&=-\int_{t}^{T}\left(F_{s}ds+F_{x}dX^{\pi}\left(s\right)+\frac{1}{2}F_{xx}\sigma^{2}\pi^{2}\left(s\right)X^{\pi}\left(s\right)^{2}ds\right)\\
&+F\left(T,X^{\pi}\left(T\right)\right).
\end{aligned}$$

Inserting (6) which for an arbitrary strategy π means that

$$F_t \leq f_t - (r + \pi(\alpha - r)) x (F_x - f_x) - \frac{1}{2}\sigma^2 \pi^2 x^2 (F_{xx} - U(f, Y^\pi, Z^\pi)),$$

with $x = X^\pi(s)$, and inserting (7) and (2) we get that

$$F(t, X^\pi(t)) \geq -\int_t^T \begin{pmatrix} \begin{pmatrix} f_s - (r + \pi(s)(\alpha - r)) X^\pi(s)(F_x - f_x) \\ -\frac{1}{2}\sigma^2 \pi(s)^2 X^\pi(s)^2 (F_{xx} - U(f, Y^\pi, Z^\pi)) \end{pmatrix} ds \\ +F_x ((r + \pi(s)(\alpha - r)) X^\pi(s) ds + \pi(s) \sigma X^\pi(s) dW(s)) \\ +\frac{1}{2} F_{xx} \sigma^2 \pi^2(s) X^\pi(s)^2 ds \end{pmatrix}$$
$$+ f(T, X^\pi(T), g(X^\pi(T)), h(X^\pi(T)))$$

Abbreviation gives

$$F(t, X^\pi(t)) \geq -\int_t^T \begin{pmatrix} \begin{pmatrix} f_s + f_x (r + \pi(s)(\alpha - r)) X^\pi(s) \\ +\frac{1}{2}\sigma^2 \pi(s)^2 X^\pi(s)^2 U(f, Y^\pi, Z^\pi) \end{pmatrix} ds \\ +F_x \pi(s) \sigma X^\pi(s) dW(s) \end{pmatrix}$$
$$+ f(T, X^\pi(T), g(X^\pi(T)), h(X^\pi(T))).$$

Inserting (53) we get that

$$F(t, X^\pi(t)) \geq \int_t^T (f_x + f_y Y_x^\pi + f_z Z_x^\pi - F_x) \pi(s) \sigma X^\pi(s) dW(s) \tag{54}$$
$$+ f(t, X^\pi(t), y^\pi(t, X^\pi(t)), z^\pi(t, X^\pi(t))).$$

Now, assuming sufficient integrability, taking conditional expectation on both sides and thereafter supremum over π on both sides finally gives

$$F(t, x) \geq \sup_\pi f(t, x, y^\pi(t, x), z^\pi(t, x)). \tag{55}$$

Consider the specific strategy π^*.

1. First, since $G(t, x) = Y^{\pi^*}(t, x)$ and $H(t, x) = Z^{\pi^*}(t, x)$ we have from (48) and (52) that

$$G(t, x) = y^{\pi^*}(t, x),$$
$$H(t, x) = z^{\pi^*}(t, x).$$

2. Second, also for this specific strategy we have that

$$f\left(t, X^{\pi^*}(t), y^{\pi^*}\left(t, X^{\pi^*}(t)\right), z^{\pi^*}\left(t, X^{\pi^*}(t)\right)\right)$$
$$= -\int_t^T \begin{pmatrix} f_s ds + f_x (r + \pi^*(s)(\alpha - r)) X^{\pi^*}(s) ds \\ + \left(f_x + f_y Y_x^{\pi^*} + f_z Z_x^{\pi^*}\right) \pi^*(s) \sigma X^{\pi^*}(s) dW(s) \\ +\frac{1}{2} U\left(f, Y^{\pi^*}, Z^{\pi^*}\right) \sigma^2 \pi^*(s)^2 X^{\pi^*}(s)^2 ds \end{pmatrix} \tag{56}$$
$$+ f\left(T, X^{\pi^*}(T), g\left(X^{\pi^*}(T)\right), h\left(X^{\pi^*}(T)\right)\right).$$

3. Third, we establish on the basis of (56) that

$$F(t, x) \geq \sup_\pi f(t, x, y^\pi(t, x), z^\pi(t, x)).$$

An Ito calculation on F gives that

$$F\left(t, X^{\pi^*}(t)\right) = -\int_t^T dF\left(s, X^{\pi^*}(s)\right) + F\left(T, X^{\pi^*}(T)\right)$$
$$= -\int_t^T \left(F_s ds + F_x dX^{\pi^*}(s) + \frac{1}{2} F_{xx} \sigma^2 \pi^*(s)^2 X^{\pi^*}(s)^2 ds\right)$$
$$+ F\left(T, X^{\pi^*}(T)\right).$$

Inserting (6) which for the strategy π^* means that

$$F_t = f_t - (r + \pi^* (\alpha - r)) x (F_x - f_x) - \frac{1}{2}\sigma^2 (\pi^*)^2 x^2 \left(F_{xx} - U\left(f, Y^{\pi^*}, Z^{\pi^*}\right)\right),$$

with $x = X^{\pi^*}(s)$, and inserting (7) and (2) with the strategy π^* we get that

$$F\left(t, X^{\pi^*}(t)\right) = -\int_t^T \left(\begin{array}{c} \left(\begin{array}{c} f_s - (r + \pi^*(s)(\alpha - r)) X^{\pi^*}(s)(F_x - f_x) \\ -\frac{1}{2}\sigma^2 \pi^*(s)^2 X^{\pi^*}(s)^2 \left(F_{xx} - U\left(f, Y^{\pi^*}, Z^{\pi^*}\right)\right) \end{array} \right) ds \\ +F_x \left(\begin{array}{c} (r + \pi^*(s)(\alpha - r)) X^{\pi^*}(s)\, ds \\ +\pi^*(s)\,\sigma X^{\pi^*}(s)\, dW(s) \end{array} \right) \\ +\frac{1}{2} F_{xx}\sigma^2 \pi^*(s)^2 X^{\pi^*}(s)^2\, ds \end{array} \right)$$
$$+ f\left(T, X^{\pi^*}(T), g\left(X^{\pi^*}(T)\right), h\left(X^{\pi^*}(T)\right)\right)$$

Abbreviation gives

$$F\left(t, X^{\pi^*}(t)\right) = -\int_t^T \left(\begin{array}{c} \left(\begin{array}{c} f_s + f_x (r + \pi^*(s)(\alpha - r)) X^{\pi^*}(s) \\ +\frac{1}{2}\sigma^2 \pi^*(s)^2 X^{\pi^*}(s)^2 U\left(f, Y^{\pi^*}, Z^{\pi^*}\right) \end{array} \right) ds \\ +F_x \pi^*(s)\,\sigma X^{\pi^*}(s)\, dW(s) \end{array} \right)$$
$$+ f\left(T, X^{\pi^*}(T), g\left(X^{\pi^*}(T)\right), h\left(X^{\pi^*}(T)\right)\right).$$

Inserting (56) we get that

$$F\left(t, X^{\pi^*}(t)\right) = \int_t^T \left(f_x + f_y Y_x^{\pi^*} + f_z Z_x^{\pi^*} - F_x\right) \pi^*(s)\,\sigma X^{\pi^*}(s)\, dW(s)$$
$$+ f\left(t, X^{\pi^*}(t), y^{\pi^*}\left(t, X^{\pi^*}(t)\right), z^{\pi^*}\left(t, X^{\pi^*}(t)\right)\right).$$

Now, assuming sufficient integrability, taking conditional expectation on both sides finally gives

$$F(t, x) = f\left(t, x, y^{\pi^*}(t, x), z^{\pi^*}(t, x)\right) \le \sup_\pi f\left(t, x, y^\pi(t, x), z^\pi(t, x)\right). \tag{57}$$

(55) together with (57) gives that

$$F(t, x) = \sup_\pi f\left(t, x, y^\pi(t, x), z^\pi(t, x)\right).$$

From the arguments above we learn that this supremum is obtained by the strategy π^*.

■