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ORIGINAL ARTICLE

Optimal control of an objective functional with non-linearity between the conditional expectations: solutions to a class of time-inconsistent portfolio problems

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Abstract
We present a modified verification theorem for the equilibrium control of a general class of portfolio problems. The general class of portfolio problems studied in this paper, is characterized by an objective where the investor seeks to maximize a functional of two conditional expectations of terminal wealth. The objective functional is allowed to be non-linear in the conditional expectations, and thus the problem class is in general terms time-inconsistent. In addition, we provide a corrected proof of the verification theorem and apply the theorem to a number of quadratic, time-inconsistent portfolio problems and determine their solutions. Some of the quadratic portfolio problems have not previously been solved analytically.

Keywords Time-inconsistency · Quadratic portfolio problems · Optimal control · Equilibrium control laws

1 Introduction

In this paper we study portfolio problems where the investor seeks to maximize the functional

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$$f(t, x, E_{t,x}[g(X^{\pi}(T))], E_{t,x}[h(X^{\pi}(T))]), \tag{1}$$

for some functions f, g, h and where t denotes time, x denotes wealth at time t and $X^{\pi}(T)$ express the stochastic terminal wealth at time T under the control π. The novelty of the problem class presented by the objective in Eq. (1), is the possibility of a non-linear structure between the two conditional expectations. The extended problem class was originally presented in Kryger and Steffensen (2010).

In the present paper, we provide a modified version of the verification theorem first introduced in Kryger and Steffensen (2010). The theorem provides verification of the equilibrium control law of the investor who wishes to dynamically maximize the functional in Eq. (1). The new representation of the theorem is expressed in terms of equilibrium control laws, a concept originally introduced in Björk and Murgoci (2010), and in addition provides a more compact pseudo-Bellman equation than the original theorem presented in Kryger and Steffensen (2010). The new theorem is simpler to apply, since we have eliminated a double-specification of the value function. In addition to the new theorem, we solve a number of quadratic investment problems, some of which have not previously been solved analytically. Finally, we provide a corrected proof of the verification theorem, following the equilibrium reasoning of Björk and Murgoci (2010). Thus, compared to the unpublished (but electronically available) paper Kryger and Steffensen (2010) we provide here a modified result including a corrected proof and offer a series of new examples.

We present the examples in a compendium-type manner in order to provide a complete picture of the structures of problems and their solutions. With this approach, we include both some examples that make little economic sense because the resulting investment strategies have undesirable properties from a practical point of view, and some (other) examples where we do not have existence and uniqueness results. In the latter cases, we only solve the problems up to a system of ODE's for which we do not provide existence and uniqueness results. These examples are included to make the compendium of problems and related solutions more complete. We discuss for each example its impact, and in concrete examples the lack of impact due to either undesirable economic properties or lack of existence and uniqueness results.

The game theoretic approach to time-inconsistent investment problems was first suggested by Strotz (1955). The approach was formally defined by Björk and Murgoci (2010) in a general Markovian framework. In traditional portfolio optimization over terminal wealth, $X^{\pi}(T)$, we are able to apply the standard dynamic programming principle and determine the investment strategy, $\pi^*(t)$, which achieves the supremum

$$\sup_{\pi} E_{t,x}[F(X^{\pi}(T))].$$

In Björk and Murgoci (2010), the optimization problem over terminal wealth was extended to cover an investor who seeks to maximize the functional

$$E_{t,x}[F(t, x, X^{\pi}(T))] + G(t, x, E_{t,x}[X^{\pi}(T)]), \tag{2}$$

(noting that Björk and Murgoci (2010) originally also includes optimization over consumption). The new objective introduces time-inconsistent preferences and Björk and Murgoci (2010) present a new formal definition of a dynamically optimal investment strategy; the equilibrium control law. As commented in Björk and Murgoci (2010), one might trivially introduce a function g inside the conditional mean of their G-function in Eq. (2). We consider the equilibrium control law as a meaningful solution to the non-standard time-inconsistent investment problem and the game theoretical foundations for interpretation of the non-standard problem is taken for given in our presentation. We refer to Björk and Murgoci (2010) for more theoretical considerations in this respect.

The objective considered in the present study covers a number of mean–variance-related portfolio problems. Portfolio optimization with time-inconsistent mean–variance preferences has been studied extensively in a variety of different settings in the last decade. The dynamic mean–variance problem was first solved informally in a backward recursion manner in an incomplete market setting in Basak and Chabakauri (2010) and as a special case of the formal disposition of Björk and Murgoci (2010). Following these two papers, the mean–variance objective has been studied in numerous different set-ups. Wang and Forsyth (2011) studied the problem including constraints to the investment policy. Zeng and Li (2011) consider mean–variance optimization of investment and reinsurance policies from the perspective of insurers. Czichowsky (2013) exploit the linear-quadratic structure of the mean–variance problem and present a formal way to handle a setting more general than the Markovian of the present paper and how to apply martingale techniques. Interestingly, Czichowsky (2013) shows that non-uniqueness of the optimal strategy can exist in markets more general than the one we study. A more general mean–variance setting is introduced in Björk et al. (2014), whereas Björk and Murgoci (2014) is devoted to a complete analysis of the problem formulation and solution in discrete time. Wu (2013) determines the equilibrium value function for the mean–variance investor in discrete time with constant and wealth-dependent risk aversion respectively. In Wei et al. (2013) the mean–variance portfolio is determined in a setting with regime switching. In Bensoussan et al. (2014) it is discovered how the discrete-time version of the mean–variance investor with wealth-dependent risk aversion leads to an unbounded value function, why they solve a discrete-time problem with the inclusion of short-selling constraints. Wu et al. (2015) studies mean–variance optimization in the context of Defined Contributions (DC) pension fund management with the inclusion of inflation risk and salary risk. Similarly, in Sun et al. (2016) the cases of the pre-committed investor as well as the equilibrium strategy are determined for a DC pension plan in a market where the risky asset is modelled by a jump-diffusion. Their results show how the behaviour of the optimally controlled wealth is fundamentally different, depending on the approach to optimization. In the present paper, we work exclusively with equilibrium strategies and the resulting wealth dynamics. A different extension to the problem class in the mean–variance setting was studied in Zhang and Liang (2016) with the inclusion of jumps in the risky asset. Bannister et al. (2016) consider the mean–standard-deviation investor in the case where the investor only has access to risky investments in a multiperiod set-up. However, time-inconsistency arise through other objectives than the mean–variance optimization. Björk et al. (2017) work with an objective similar to the one by Björk and Murgoci (2010) and provide new examples, including one with a general market equilibrium, such that there are actually two 'dimensions' of the equilibrium, intrapersonal (but intertemporal) equilibrium, and (interpersonel among agents) market equilibrium in a usual asset pricing theoretical sense. Hu et al. (2017) study time-inconsistent linear-quadratic problems with stochastic (market) coefficients. Their complete market example is a special case of our example called generalized mean–variance criterion. In Ekeland et al. (2012) the inclusion of life-insurance and different utility of the investor and heirs in combination with hyperbolic discounting is considered. In Kronborg and Steffensen (2015) a problem similar to the one in the present paper, but with the inclusion of consumption and labour income and restricted to the functions $g(x) = x$ and $h(x) = x^2$, is presented.

The main contribution of the present paper is to present a new representation and proof of the verification result for the equilibrium strategy of the extended problem class, defined by the objective in Eq. (1). In addition, we present the solutions to a broad class of portfolio problems with quadratic objectives, some to which existing mean–variance optimization results are special cases and some new objectives with non-linearity in the conditional expectations. We also present the repeated structure of the analytical solutions to portfolio problems with quadratic objectives.

2 The main result

We consider a market consisting of a bond and a stock with dynamics given by

$$\begin{aligned} dB(t) &= rB(t)\,dt,\; B(0) = 1, \\ dS(t) &= \alpha S(t)\,dt + \sigma S(t)\,dW(t),\; S(0) = s_0 > 0, \end{aligned}$$

with $r < \alpha$ and $\alpha,\ \sigma > 0$. W is a standard Brownian motion on an abstract probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions; and with each $\mathcal{F}_t \subseteq \sigma\{W(s),\ 0 \leq s \leq t\}$. Further we define $\theta = (\alpha - r)/\sigma$, the market price of risk.

We consider an investor, who places the proportion $\pi(t)$ of his wealth in the stock at time t. Denoting by $X^\pi(t)$ his wealth at time t given the investment strategy π, the dynamics of his wealth becomes

$$\begin{aligned} dX^\pi(t) &= (r + \pi(t)(\alpha - r))\,X^\pi(t)\,dt + \pi(t)\,\sigma X^\pi(t)\,dW(t), \\ X^\pi(0) &= x_0 > 0, \end{aligned}$$

where x_0 is the initial wealth. The strategy is self-financing in the sense that we disregard consumption and injection of capital.

Before introducing the objective, we introduce two conditional expectations

$$y^\pi(t,x) := E_{t,x}\left[g\left(X^\pi(T)\right)\right], \tag{3}$$

$$z^\pi(t,x) := E_{t,x}\left[h\left(X^\pi(T)\right)\right], \tag{4}$$

for functions g and h. The subscript t, x denotes conditioning on the event $X^{\pi}(t) = x$.

The objective of the investor, considered in this study, is to maximize the value function

$$J^{\pi}(t,x) := f\left(t, x, y^{\pi}(t,x), z^{\pi}(t,x)\right), \tag{5}$$

defined through a given regular function $f \in C^{1,2,2,2}$, and to find the corresponding optimal investment strategy, π^*. We denote the *optimal* value function by $V(t,x) := J^{\pi^*}(t,x)$. As opposed to Björk and Murgoci (2010) we only treat problems over terminal wealth. Also, we restrict ourselves to the (one-dimensional) Black–Scholes market.

The portfolio problem presented in Eq. (5) is, in its general form, not a classical time consistent portfolio problem. If we fix a point in time, (t_0, x_0), and determine the future investment strategy, $\bar{\pi}_0(t,x)$ for all $(t,x) \in [t_0, T] \times \mathbb{R}$, which maximizes the value function today, in the sense that $J^{\bar{\pi}_0}(t_0, x_0) = \sup_{\pi} J^{\pi}(t_0, x_0)$, then, when we at a later point in time, (t_1, x_1), perform the same exercise and determine the strategy $\bar{\pi}_1(t,x)$ for all $(t,x) \in [t_1, T] \times \mathbb{R}$, which achieves $J^{\bar{\pi}_1}(t_1, x_1) = \sup_{\pi} J^{\pi}(t_1, x_1)$, we do not (necessarily) find that $\bar{\pi}_0(t,x) = \bar{\pi}_1(t,x)$ for all $(t,x) \in [t_1, T] \times \mathbb{R}$. In conclusion, the (future) strategy which is optimal when maximizing the objective today is not necessarily optimal tomorrow. We have a time-inconsistent investment problem.

If f does not depend on $(t, x, z^{\pi}(t,x))$ and is affine in $y^{\pi}(t,x)$, the problem can be written in a classical way, and the objective of the investor is to determine

$$\sup_{\pi} E_{t,x}\left[g\left(X^{\pi}(T)\right)\right] + \text{ constant.} \tag{6}$$

In this case, we are able to determine a time consistent investment strategy which achieves the supremum, since we are able to exploit the law of iterated expectations and thus the Bellman optimality principle applies.

Due to the time-inconsistent preferences specified through the objective in Eq. (5), it is in the general case not sufficient to determine the (current) investment strategy that will reach the true supremum of the value function. As explained above, the investor continuously deviates from this strategy and thus does not actually achieve any of the determined supremums (in expectation). Instead, we seek to determine the equilibrium control law, as introduced in Björk and Murgoci (2010). We state the definition here and denote by $\mathcal{U} \in \mathbb{R}$ the set for which the controls π_h defined below are admissible in the sense of Definition 2.

Definition 1 *(Equilibrium control law)* Consider a control law $\hat{\pi}$ (informally viewed as a candidate equilibrium law). Choose a fixed $\pi \in \mathcal{U}$ and a real number $h > 0$. Also, fix an arbitrarily chosen initial point (t, x). Define the control law $\hat{\pi}_h$ by

$$\hat{\pi}_h(s,y) = \begin{cases} \pi & \text{for } (s,y) \in [t, t+h) \times \mathbb{R}, \\ \hat{\pi}(s,y) & \text{for } (s,y) \in [t+h, T] \times \mathbb{R}. \end{cases}$$

If

$$\liminf_{h \to 0} \frac{J^{\hat{\pi}}(t,x) - J^{\hat{\pi}_h}(t,x)}{h} \geq 0,$$

for all $\pi \in \mathcal{U}$, then we say that $\hat{\pi}$ is an equilibrium control law.

Remark 1 By definition of π_h^* we have the relation

$$J^{\pi^*}(t+h,x) = J^{\pi_h^*}(t+h,x).$$

As a result, we may write

$$\begin{aligned}
&\frac{J^{\pi^*}(t,x) - J^{\pi_h^*}(t,x)}{h} \\
&\quad = \frac{J^{\pi^*}(t,x) - J^{\pi^*}(t+h,x) - (J^{\pi_h^*}(t,x) - J^{\pi_h^*}(t+h,x))}{h} \\
&\quad = J_t^{\pi_h^*} - J_t^{\pi^*} + o(h).
\end{aligned}$$

With the equilibrium control law, we determine the investment strategy which at any point in time (t,x) maximizes the present objective of the investor, under the restriction that the future strategy is taken for given. The strategy is determined through backward recursion and in the rest of the paper we refer to the equilibrium control law as the optimal control.

The problem in Eq. (5) is, at first glance, just a mathematical abstract generalization of the problem (6). However, as we argue below, there are examples of this generalization that make good economic sense. Truly, there are also examples of (5) that make no economic sense. But this is not an argument against solving (5) in its generality, as long as we have some interesting and useful applications in mind.

Here we present a list of such motivating examples, which we solve in Sects. 4.1–4.5. The examples are presented through the objective the investor wishes to maximize in the equilibrium sense introduced above.

1. Generalized mean–variance optimization

$$(l_1(t)x + l_2(t))\, E_{t,x}\left[X^\pi(T)\right] - \frac{1}{2}E_{t,x}\left[\left(X^\pi(T) - k(t)E_{t,x}\left[X^\pi(T)\right]\right)^2\right],$$

for some functions l_1, l_2 and k.
There is additivity in the terms involving the first and second order moment and if $l_1(t) = 0$ and l_2 and k do not depend on t, then the functional the investor seeks to maximize does not depend on (t,x). There is linearity in the second order moment, but the non-linearity in the first order moment makes the problem non-standard. This is the problem treated by Basak and Chabakauri (2010) in an incomplete market framework. The objective is studied as a special (the simplest) case by Björk and Murgoci (2010). The case of $l_1(t) = \frac{1}{\gamma}$ for some constant $\gamma > 0$ and $l_2(t) = 0$ is investigated by Björk et al. (2014). The complete market example provided by Hu et al. (2017) corresponds to letting l_1 and l_2 be constants and $k = 1$.

2. Endogenous habit formation mean–variance style

$$(l_1(t)x + l_2(t))\, E_{t,x}\left[X^{\pi}(T)\right] - \frac{1}{2} E_{t,x}\left[\left(X^{\pi}(T) - xk(t)\right)^2\right],$$

   for some functions l_1, l_2, k.
   We provide the full solution to this problem although the case can also be covered using the verification theorem of Björk and Murgoci (2010).
3. Generalized mean–standard-deviation optimization

$$l_2(t) E_{t,x}[X(T)] - \left(E_{t,x}\left[\left(X^{\pi}(T) - k(t) E_{t,x}\left[X^{\pi}(T)\right]\right)^2\right]\right)^{\frac{1}{2}},$$

   for some functions l_2, k.
4. Endogenous habit formation mean–standard-deviation style

$$l_2(t) E_{t,x}\left[X^{\pi}(T)\right] - \left(E_{t,x}\left[\left(X^{\pi}(T) - xk(t)\right)^2\right]\right)^{\frac{1}{2}}.$$

   for some functions l_2, k.
5. Scaled mean–variance optimization

$$l_2(t) E_{t,x}\left[X^{\pi}(T)\right] - \frac{1}{2} \frac{Var_{t,x}\left[X^{\pi}(T)\right]}{E_{t,x}\left[X^{\pi}(T)\right]},$$

   for some function l_2.

Due to the non-additivity in the terms involving the first and second order moment in the objective functional of the investor presented in points 3., 4., and 5., these cases are not covered by the verification result of Björk and Murgoci (2010). One can come up with several other interesting examples, e.g. collective utility of heterogeneous investors Kryger and Steffensen (2010), but the above cases are the only ones studied in the present paper.

The result that facilitates the solution of this new class of problems is presented in the following theorem. First however, we require a definition of an admissible strategy to which the value function of the objective exists.

Definition 2 *(Admissibility)* Consider an arbitrary strategy π. If there exists functions $Y^{\pi}(t,x), Z^{\pi}(t,x) \in C^{1,2}$ such that

$$Y^{\pi}_t = -\left(r + \pi(\alpha - r)\right) x Y^{\pi}_x - \frac{1}{2}\sigma^2 (\pi)^2 x^2 Y^{\pi}_{xx}, \tag{7}$$

$$Y^{\pi}(T,x) = g(x), \tag{8}$$

and

$$Z^{\pi}_t = -\left(r + \pi(\alpha - r)\right) x Z^{\pi}_x - \frac{1}{2}\sigma^2 \left(\pi^*\right)^2 x^2 Z^{\pi}_{xx}, \tag{9}$$

$$Z^{\pi}(T, x) = h(x), \tag{10}$$

and such that the processes

$$\sigma\pi(s)X^{\pi}(s)Y_x^{\pi}(s, X^{\pi}(s)), \tag{11}$$
$$\sigma\pi(s)X^{\pi}(s)Z_x^{\pi}(s, X^{\pi}(s)) \tag{12}$$

are in $\mathcal{L}^2$ and such that the function $\bar{f}(t, x) := f(t, x, Y^{\pi}(t, x), Z^{\pi}(t, x))$ is in $C^{1,2}$, then the strategy π is called admissible w.r.t. the functions $Y^{\pi}(t, x)$ and $Z^{\pi}(t, x)$.

Theorem 1 *Let $f : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. Let g and h be real functions. The set of* admissible *strategies are given by Definition 2. Note that admissibility depends on the choice of g, h.*

Consider the investor with value function

$$J^{\pi}(t, x) := f\left(t, x, y^{\pi}(t, x), z^{\pi}(t, x)\right)$$

with

$$y^{\pi}(t, x) = E_{t,x}\left[g\left(X^{\pi}(T)\right)\right],$$
$$z^{\pi}(t, x) = E_{t,x}\left[h\left(X^{\pi}(T)\right)\right].$$

Denote by $V(t, x)$ the optimal value function in the sense of an equilibrium control law of Definition 1.

If there exist two functions G, $H \in C^{1,2}$ such that the control law

$$\pi^* = \arg\inf_{\pi}\left[-(r + \pi(\alpha - r))x\left(f_y G_x + f_z H_x\right) - \frac{1}{2}\sigma^2\pi^2x^2\left(f_y G_{xx} + f_z H_{xx}\right)\right], \tag{13}$$

is an admissible strategy w.r.t. G, and H, then

$$y^{\pi^*}(t, x) = G(t, x), z^{\pi^*}(t, x) = H(t, x),$$

and the optimal investment strategy is given by π^*, and the optimal value function is determined by

$$V(t, x) = f(t, x, G(t, x), H(t, x)). \tag{14}$$

Proof See the "Appendix".

We find the optimizing investment strategy in terms of the functions f, G and H by differentiating with respect to π inside the square brackets of Eq. (13) and get

$$\pi^* = -\frac{\alpha - r}{\sigma^2 x} \cdot \frac{f_y G_x + f_z H_x}{f_y G_{xx} + f_z H_{xx}}, \tag{15}$$

providing $(f_y G_{xx} + f_z H_{xx}) < 0$.

Remark 2 The theorem can easily be extended to cover more than two transformations of terminal wealth:

$$J^{\pi}(t,x) = f\left(t, x, y_1^{\pi}(t,x), \ldots, y_n^{\pi}(t,x)\right),$$

for $y_i^{\pi} = E_{t,x}\left[g_i\left(X^{\pi}(T)\right)\right]$.

Remark 3 The standard case can be formalized by

$$f(t,x,y,z) = y.$$

Then the result collapses into a standard Bellman equation. This is concluded since

$$f_t = f_x = f_z = 0,$$

and the differential equation for G in this case corresponds to the traditional Bellman equation since $V = G$.

Remark 4 We immediately observe that the optimal value function V, specified in Eq. (14) of Theorem 1, satisfies the pseudo-Bellman equation

$$V_t - f_t = \inf_{\pi}\left[-(r + \pi(\alpha - r))\,x\left(f_y G_x + f_z H_x\right) - \frac{1}{2}\sigma^2\pi^2x^2\left(f_y G_{xx} + f_z H_{xx}\right)\right]. \tag{16}$$

This is a direct result of differentiating V by use of the chain-rule and the fact that G and H satisfy the system of PDE's given in Eqs. (7)–(10) with π^* defined in Eq. (13). A similar representation is found in Björk et al. (2014) in the special case of mean–variance optimization with t-dependent γ.

Remark 5 We further observe that the pseudo-Bellman equation in (16) can be written as

$$V_t - f_t = \inf_{\pi}\left[-(r + \pi(\alpha - r))\,x\,(V_x - f_x) - \frac{1}{2}\sigma^2\pi^2x^2\,(V_{xx} - U)\right],$$

with $U := f_{xx} + 2f_{xy}G_x + 2f_{xz}H_x + f_{yy}G_x^2 + 2f_{yz}G_xH_x + f_{zz}H_x^2$. This representation corresponds to the pseudo-Bellman equation originally presented in Kryger and Steffensen (2010). The representation is a direct consequence of the chain-rule applied to $V(t,x) = f(t,x,G(t,x),H(t,x))$.

Remark 6 We finally observe that in comparison the pseudo-Bellman equation introduced in Björk and Murgoci (2010) (for the case without consumption) we would have $f_z = 1$ and in addition $f_{xz} = f_{yz} = f_{zz} = 0$.

In the next sections, we solve the problems listed above. First, we observe that the structure of the solutions repeats itself. Therefore, in Section 3, we first introduce the solution structure in general terms. Next, we present the specific solutions to the cases listed above in Sect. 4.

3 Repeated structure of the solutions

In Sect. 4 we present the solutions to the problems from the list in Sect. 2. In all of the cases we have the functions $g(x) = x$ and $h(x) = x^2$ and we search for a solution in the form

$$G(t, x) = a(t)x + b(t), \tag{17}$$

$$H(t, x) = c(t)x^2 + d(t)x + e(t), \tag{18}$$

with terminal conditions $a(T) = c(T) = 1$ and $b(T) = d(T) = e(T) = 0$ and the requirement $H(t, x) \geq 0$. The partial derivatives are

$$G_t = a_t x + b_t,\ \ G_x = a(t),\ \ G_{xx} = 0,$$

$$H_t = c_t x^2 + d_t x + e_t,\ \ H_x = 2c(t)x + d(t),\ \ H_{xx} = 2c(t).$$

With this guess, the optimal investment candidate from Eq. (15) becomes

$$\pi^*(t, x)x = -\frac{\alpha - r}{\sigma^2}\left(\frac{1}{2}\frac{f_y}{f_z}\frac{a(t)}{c(t)} + x + \frac{1}{2}\frac{d(t)}{c(t)}\right), \tag{19}$$

where the partial derivatives of f have arguments $(t, x, G(t, x), H(t, x))$ In the following cases, our guess is always covered by the specification in Eqs. (17)–(18) and thus we return to the general specification of π^* in Eq. (19) again and again. In order for $\pi^*(t)$ in Eq. (19) to realize the infimum in Eq. (13), we must have $f_z 2c(t) < 0$. This condition is checked in all of the following cases.

3.1 Special case: when $\pi^*(t, x)x$ is affine in x

In addition, we highlight the general case where the optimal strategy from Eq. (19) turns out to be in the form

$$\pi^*(t, x)x = p_1(t)x + p_2(t), \tag{20}$$

when the partial derivatives f_y, f_z are inserted. When plugging the optimal strategy from Eq. (20) and the partial derivatives corresponding to the general guess of Eqs. (17)–(18) into the PDE's in Eqs. (7) and (9) of Definition 2, we see that it is possible to separate the resulting system into a highly non-linear system of ODE's by isolating the different x-dependences. We arrive at

$$a_t = -\,[r + p_1(t)(\alpha - r)]a(t), \tag{21}$$

$$b_t = -\,p_2(t)(\alpha - r)a(t), \tag{22}$$

$$c_t = -\,[2r + 2(\alpha - r)p_1(t) + \sigma^2 p_1^2(t)]c(t), \tag{23}$$

$$d_t = -\,[r + (\alpha - r)p_1(t)]d(t) - [2(\alpha - r)p_2(t) + 2\sigma^2 p_1(t)p_2(t)]c(t), \tag{24}$$

$$e_t = -\,(\alpha - r)p_2(t)d(t) - \sigma^2 c(t)p_2^2(t), \tag{25}$$

with terminal conditions $a(T) = c(T) = 1$ and $b(T) = d(T) = e(T) = 0$. We note that the functions p_1 and p_2 themselves have to be determined and are expressed in terms of a, b, c, d and e. Therefore, the seemingly linear structure of (21)–(25) is, actually, in general highly non-linear and becomes only linear in a few concrete cases. Existence and uniqueness of an optimal control is, in each case studied below, essentially equivalent to existence and uniqueness of a solution to the—generally non-linear—system of ODEs, (21)–(25). To arrive at the system (21)–(25) it is simply required that $\pi^*(t,x)$ can be specified as in Eq. (20) for some functions p_1, p_2. As it turns out, this structure is repeated in all of the cases covered in Sects. 4.1–4.5.

3.1.1 Solution when $\pi^*(t,x)x = p_2(t)$

It is immediately seen, that if $p_1(t) = 0$, then the system in Eqs. (21)–(25) reduces to having solution $c(t) = a^2(t)$ and $d(t) = 2a(t)b(t)$ with

$$
\begin{aligned}
a_t &= -ra(t), \;\; a(T) = 1, \\
b_t &= -p_2(t)(\alpha - r)a(t), \;\; b(T) = 0, \\
e_t &= -2(\alpha - r)p_2(t)a(t)b(t) - \sigma^2 p_2^2(t)a^2(t), \;\; e(T) = 0.
\end{aligned}
$$

By defining $\varepsilon(t) := e(t) - b^2(t)$ we are able to write the system even more compactly as

$$b_t = -p_2(t)(\alpha - r)a(t), \;\; b(T) = 0, \tag{26}$$

$$\varepsilon_t = -\sigma^2 p_2^2(t)a^2(t), \;\; \varepsilon(T) = 0, \tag{27}$$

with

$$a(t) = e^{r(T-t)}. \tag{28}$$

As a result, we conclude that whenever the candidate for the optimal investment strategy can be written in the form in Eq. (20), and when the guess $c(t) = a^2(t)$ and $d(t) = 2a(t)b(t)$ results in $p_1(t) = 0$, then the optimal investment strategy is determined by solving the ODE system in Eqs. (26)–(28). We recall that in the special case of this section, we have $\pi^*(t,x)x = p_2(t)$ by Eq. (20).

The above system can thus also be expressed in terms of the optimal strategy by

$$
\begin{aligned}
b_t &= -\pi^*(t,x)x(\alpha - r)a(t), \;\; b(T) = 0, \\
\varepsilon_t &= -\sigma^2(\pi^*(t,x)x)^2 a^2(t), \;\; \varepsilon(T) = 0.
\end{aligned}
$$

3.1.2 Solution when $\pi^*(t,x)x = p_1(t)x$

It is immediately seen, that if $p_2(t) = 0$, then $b(t) = e(t) = 0$ and the system in Eqs. (21)–(25) reduces to

$$a_t = -[r + \pi^*(t)(\alpha - r)]a(t), \;\; a(T) = 1, \tag{29}$$

$$c_t = -\left[2r + 2(\alpha - r)\pi^*(t) + \sigma^2(\pi^*(t))^2\right]c(t),\ \ c(T) = 1, \tag{30}$$

and in addition

$$d_t = -\left[r + (\alpha - r)\pi^*(t)\right]d(t),\ \ d(T) = 0,$$

where the guess $d(t) = 0$ is a solution. As a result, we conclude that whenever the candidate for the optimal investment strategy can be written in the form in Eq. (20), and when the guess $b(t) = e(t) = d(t) = 0$ results in $p_2(t) = 0$, then the optimal investment strategy is simply determined by solving the ODE system in Eqs. (29)–(30).

4 Cases

In this section, we present the solutions to the problems presented in the list above.

4.1 Generalized mean–variance

In this subsection, we consider the optimization problem where the investor wishes to maximize

$$(l_1(t)x + l_2(t))\, E_{t,x}\left[X^\pi(T)\right] - \frac{1}{2}E_{t,x}\left[\left(X^\pi(T) - k(t)E_{t,x}\left[X^\pi(T)\right]\right)^2\right],$$

for some functions l_1, l_2 and k. The solution to this problem, when $k(t) = 1$ and $l_1(t) = 0$ and $l_2(t) = \frac{1}{\gamma}$, was found by Basak and Chabakauri (2010) in a relatively general incomplete market. Björk and Murgoci (2010) also give the solution as an example of their extended HJB equation. The solution when $k(t) = 1$ and $l_1(t) = \frac{1}{\gamma}$ and $l_2(t) = 0$ was derived in Björk et al. (2014). We retrieve their results as special cases in the following subsections.

In the general form, the problem is specified through the f-function

$$f(t, x, y, z) = (l_1(t)x + l_2(t))\, y - \frac{1}{2}\left(z - (2 - k(t))k(t)y^2\right), \tag{31}$$

$$f_y = (l_1(t)x + l_2(t)) + (2 - k(t))k(t)y, \tag{32}$$

$$f_z = -\frac{1}{2}, \tag{33}$$

and functions $g(x) = x$ and $h(x) = x^2$.

We search for a solution in the form

$$G(t, x) = a(t)x + b(t),$$

$$H(t, x) = c(t)x^2 + d(t)x + e(t).$$

We recall that the solution requires $H(t,x) \geq 0$, for the sample space of the wealth, resulting from the optimal investment strategy. The candidate optimal strategy is determined by plugging the partial derivatives of f from Eqs. (32)–(33) into Eq. (19). We arrive at a strategy in the general form specified in Eq. (20) with

$$p_1(t) = \frac{\alpha - r}{\sigma^2}\left(l_1(t)\frac{a(t)}{c(t)} + (2-k(t))k(t)\frac{a^2(t)}{c(t)} - 1\right), \quad (34)$$

$$p_2(t) = \frac{\alpha - r}{\sigma^2}\left(l_2(t)\frac{a(t)}{c(t)} + (2-k(t))k(t)\frac{a(t)b(t)}{c(t)} - \frac{1}{2}\frac{d(t)}{c(t)}\right). \quad (35)$$

In addition, we see that $\pi^*(t)$ realizes the infimum of Eq. (13), providing $-c(t) < 0$.

4.1.1 Mean–variance ($l_1(t) = 0, l_2(t) = \frac{1}{\gamma}$ and $k(t) = 1$)

In the case with $l_1(t) = 0$, $l_2(t) = \frac{1}{\gamma} > 0$ and $k(t) = 1$, the investor wishes to maximize the mean–variance objective

$$E_{t,x}\left[X^\pi(T)\right] - \frac{\gamma}{2}Var_{t,x}\left[X^\pi(T)\right],$$

and we have

$$p_1(t) = \frac{\alpha - r}{\sigma^2}\left(\frac{a^2(t)}{c(t)} - 1\right), \quad (36)$$

$$p_2(t) = \frac{\alpha - r}{\sigma^2}\left(\frac{1}{\gamma}\frac{a(t)}{c(t)} + \frac{a(t)b(t)}{c(t)} - \frac{1}{2}\frac{d(t)}{c(t)}\right). \quad (37)$$

We immediately see, that if $c(t) = a^2(t)$, then the optimal strategy specified by Eq. (20) becomes

$$\pi^*(t,x)x = p_2(t), \quad (38)$$

and thus we are in the case of Sect. 3.1.1. In conclusion, $d(t) = 2a(t)b(t)$ is a solution and

$$p_2(t) = \frac{\alpha - r}{\sigma^2\gamma}\frac{1}{a(t)}.$$

Plugging the p_2-function into Eq. (26)–(28) we finally have

$$a(t) = e^{r(T-t)},$$

and

$$b_t = -\frac{(\alpha - r)^2}{\gamma\sigma^2}, \quad b(T) = 0,$$

$$\varepsilon_t = -\frac{(\alpha - r)^2}{\gamma^2\sigma^2}, \quad \varepsilon(T) = 0,$$

with solutions

$$b(t) = \frac{\theta^2}{\gamma}(T - t),$$
$$\varepsilon(t) = \frac{\theta^2}{\gamma^2}(T - t),$$

and the optimal investment strategy in (38) becomes

$$\pi^*(t, x)\, x = \frac{\alpha - r}{\gamma\sigma^2} e^{-r(T-t)}.$$

We see that the condition $-c(t) < 0$ holds, since $c(t) = a^2(t)$, and thus the strategy does in fact realize the infimum of Eq. (13). This verifies the result of Basak and Chabakauri (2010) and Björk and Murgoci (2010). We note that $\pi^*(T, x)x = \frac{\alpha - r}{\gamma\sigma^2}$.

In this case, we do have existence and uniqueness of the optimal control. The strategy is specified as a time-dependent amount invested in stocks. Note here carefully that it is the very combination of the market structure and the objective that secures uniqueness in this case. The same objective can actually lead to non-uniqueness of the optimal strategy in more general markets, as exemplified by Czichowsky (2013). Thus, going beyond the simple market we consider can in itself destroy the uniqueness obtained in the present case and presumably also elsewhere. A strategy which is independent of wealth has little impact from a practical point of view. This is a property and a drawback that this strategy shares with investment decisions under exponential utility preferences. Note however, that exponential utility is often the preferred specification of preferences in case of utility indifference pricing because the independence of wealth is a desirable property in incomplete market pricing. In terms of risk measures, this desirable property is the same as the translation (or cash) invariance of the entropic risk measure which corresponds to the certainty equivalent of the exponential utility function, see e.g. Barrieu and El Karoui (2008). Analogously, this case of ours may be preferable to others when using mean–variance objectives for utility indifference pricing. The strategy is illustrated in a numerical example in Fig. 1 for which the amount is increasing, arriving at an amount equal to the Merton proportion at time T.

4.1.2 Scaled mean–variance ($l_1(t) = \frac{1}{\gamma}$, $l_2(t) = 0$ and $k(t) = 1$)

In the case where $l_1(t) = \frac{1}{\gamma} > 0$, $l_2(t) = 0$ and $k(t) = 1$, the investor seeks to maximize the scaled mean–variance objective

$$E_{t,x}\left[X^{\pi}(T)\right] - \frac{\gamma}{2}\frac{Var_{t,x}\left[X^{\pi}(T)\right]}{x}.$$

We see that p_1 and p_2 from Eqs. (34)–(35) become

$$p_1(t) = \frac{\alpha - r}{\sigma^2}\left(\frac{1}{\gamma}\frac{a(t)}{c(t)} + \frac{a^2(t)}{c(t)} - 1\right),$$

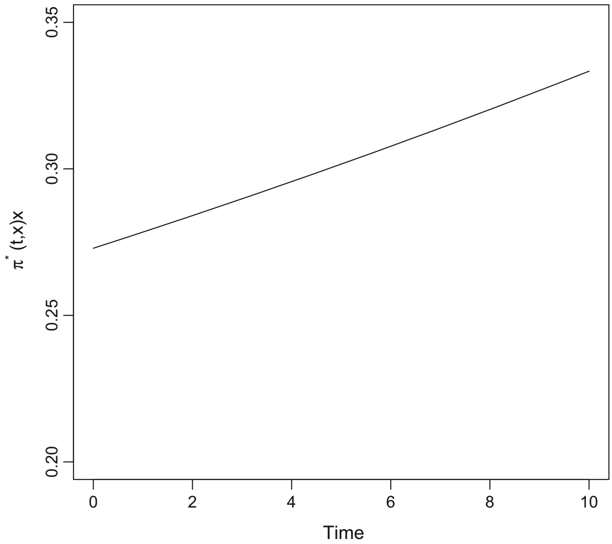

Fig. 1 Optimal amount invested in stocks, $\pi^*(t,x)x$, for the mean–variance investor with constant $\gamma = 3$ and Black–Scholes market with $r = 0.02$, $\alpha = 0.06$ and $\sigma = 0.2$ and investment horizon $T = 10$

$$p_2(t) = \frac{\alpha - r}{\sigma^2}\left(\frac{a(t)b(t)}{c(t)} - \frac{1}{2}\frac{d(t)}{c(t)}\right).$$

Guessing now that the solution has $b(t) = d(t) = 0$, we have

$$\pi^*(t) = p_1(t) = \frac{\alpha - r}{\sigma^2}\left[\frac{a^2(t)}{c(t)} - 1 + \frac{1}{\gamma}\frac{a(t)}{c(t)}\right], \tag{39}$$

and $p_2(t) = 0$ and we see that we are in the case of Sect. 3.1.2, with also $e(t) = 0$. Plugging the optimal strategy $\pi^*(t)$ from Eq. (39) into the system in Eqs. (29)–(30) gives

$$a_t = -ra(t) - \frac{(\alpha - r)^2}{\sigma^2 \gamma c(t)}\Big[\gamma[a^2(t) - c(t)] + a(t)\Big]a(t), \ \ a(T) = 1, \tag{40}$$

$$\begin{aligned} c_t = & -r2c(t) - \frac{(\alpha - r)^2}{\sigma^2\gamma}2\Big[\gamma[a^2(t) - c(t)] + a(t)\Big] \\ & - \frac{(\alpha - r)^2}{\sigma^2\gamma^2}\frac{1}{c(t)}\Big[\gamma[a^2(t) - c(t)] + a(t)\Big]^2, \ \ c(T) = 1, \end{aligned} \tag{41}$$

corresponding to the result of Björk et al. (2014). The t-derivative of the optimal strategy $\pi^*(t) = p_1(t)$ is

$$\pi^*_t = \sigma^2(\pi^*(t))^3 + (\alpha - r)(\pi^*(t))^2 + \frac{\alpha - r}{\sigma^2}\frac{1}{\gamma}\frac{a(t)}{c(t)}[r + (\alpha - r)\pi^*(t)], \tag{42}$$

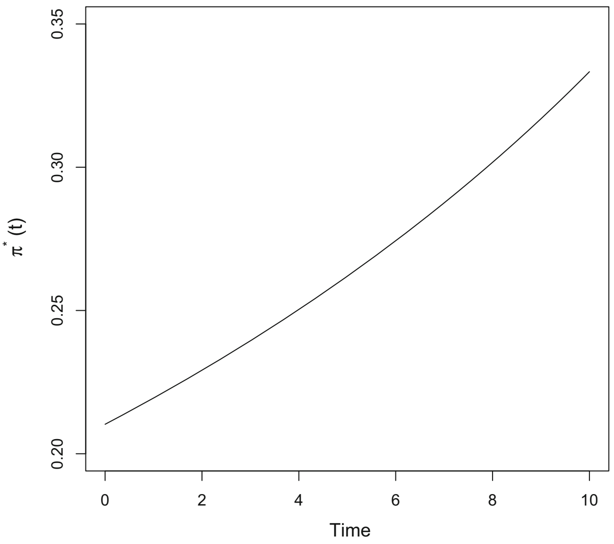

Fig. 2 Optimal proportion of wealth invested in stocks, $\pi^*(t)$, for the scaled mean–variance investor with $\gamma = 3$ and Black–Scholes market with $r = 0.02$, $\alpha = 0.06$ and $\sigma = 0.2$ and investment horizon $T = 10$

with boundary condition $\pi^*(T) = \frac{\alpha - r}{\sigma^2 \gamma}$. As a result, we are not immediately able to find the optimal strategy by solving a single ODE, since there remains a dependence on $\frac{a(t)}{c(t)}$. Rather, we must solve a system of two ODE's, e.g. Eqs. (40)–(41) and plugging the solutions into Eq. (39). This is not a problem numerically, whenever a solution exists. We refer to Björk et al. (2014) for a proof of existence and uniqueness of a solution $\pi^*(t)$ when $\alpha - r > 0$ and $\gamma > 0$. We also note, that for a given solution we must check that $-\frac{\gamma}{x}c(t) < 0$ for all (x, t) to ensure we have achieved the maximum. This is generally fulfilled if $\gamma > 0$ and $c(t) > 0$ for all t, while the wealth of the investor x remains positive throughout the investment horizon. Since the investment strategy is only t-dependent, the wealth process $X^{\pi^*}(t)$ remains positive as it is a generalized Geometric Brownian Motion (GBM).

In this case, we do have existence and uniqueness of the optimal control, following from Björk et al. (2014). The strategy is specified as a time-dependent proportion invested in stocks. A specification in terms of a time-dependent proportion is desirable from a practical point of view. This is a property shared with investment decisions under constant relative risk aversion. One can think of several practical and theoretical reasons why the proportion should be decreasing over the time horizon. The strategy is illustrated in a numerical example in Fig. 2 for which the proportion is, however, increasing, terminating in the Merton proportion at time T. Note though, that the monotonicity of the optimal control can be varied by introducing time-dependence of l_1 and k. Time-dependence of l_1 is here equivalent with time-dependence of γ. One has to be careful with comparing that with a Merton problem with time-dependent relative risk aversion which, to the knowledge of the authors, has not been studied from an equilibrium point of view, only in a classical consumption-investment setting, see Steffensen (2011). Typically, a decreasing proportion is motivated by problem features, that we have not taken into account here, e.g. labor income. Thus, an increasing strategy in our example may just be considered as a dampening effect on an otherwise decreasing proportion coming from other features of more realistic setting. Of course, one would have to formulate and solve the problem with labor income to see which of the two opposite effects really dominates for realistic parameters. This, on the other hand, would re-introduce wealth-dependence of the optimal proportion.

It is clear from the two strategies studied in Sect. 4.1.1 and the present Sect. 4.1.2, that the equilibrium control, and therefore the controlled process, are fundamentally different depending on the specific structure of the mean–variance objective. Further, for each problem, one may compare with the optimally controlled process if that same objective had been considered by a pre-committed investor. E.g., for scaled mean–variance in the present subsection, the optimally controlled process is a geometric Brownian motion. In case of pre-commitment, the problem is essentially a classical linear-quadratic problem with the optimally controlled wealth being a mean-reverting process. Thus, it appears that pre-commitment imposes some kind of stability of the wealth dynamics that does not hold for the equilibrium case that lead to exponential growth of wealth. However, it is beyond the scope of this paper to examine whether this observation is something special for this case of scaled mean–variance or if this is something that in some sense holds in generality.

4.1.3 Endogenous mean-habit ($l_1(t) = l_2(t) = 0$)

In this case the optimization problem corresponds to the investor with the objective to minimize

$$\frac{1}{2}E_{t,x}\left[\left((X^{\pi}(T) - k(t)E_{t,x}\left[X^{\pi}(T)\right]\right)^2\right].$$

The problem is an adjustment to the traditional endogenous habit formation optimization, where the habit formation is expressed in terms of the current conditional mean instead of the current level of wealth. To our knowledge, the result is new.

We immediately see that the functions p_1, p_2 from Eqs. (34)–(35) become

$$\begin{aligned} p_1(t) =& \frac{\alpha - r}{\sigma^2}\left((2 - k(t))k(t)\frac{a^2(t)}{c(t)} - 1\right), \\ p_2(t) =& \frac{\alpha - r}{\sigma^2}\left((2 - k(t))k(t)\frac{a(t)b(t)}{c(t)} - \frac{1}{2}\frac{d(t)}{c(t)}\right), \end{aligned}$$

and as a result we see that with the guess $b(t) = d(t) = 0$ we are again in the case of Sect. 3.1.2 with

$$\pi^*(t) = p_1(t) = \frac{\alpha - r}{\sigma^2}\left((2 - k(t))k(t)\frac{a^2(t)}{c(t)} - 1\right),$$

where a, c are determined as solutions to Eqs. (29)–(30). We see that the strategy realizes the infimum in Eq. (13) providing $c(t) > 0$ for all t. We have terminal value $\pi^*(T) = \frac{\alpha - r}{\sigma^2}[(2 - k(T))k(T) - 1]$. We see that the derivative is

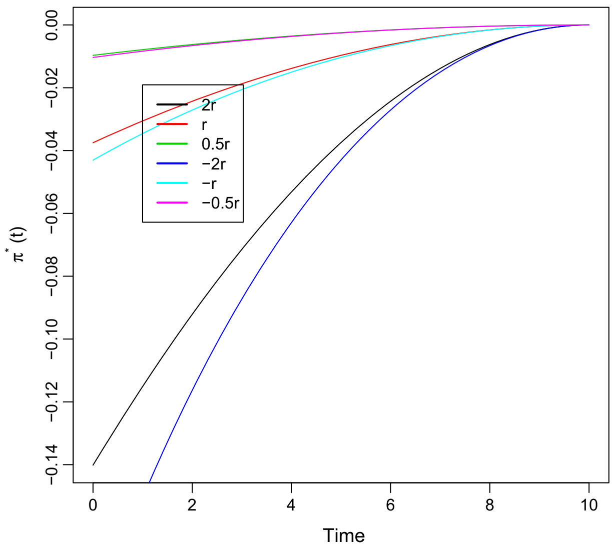

Fig. 3 Optimal proportion of wealth invested in stocks, $\pi^*(t)$, for the endogenous mean-habit investor with $k(t) = e^{\pm\bar{r}(T-t)}$ for $\bar{r} \in \{2r, r, 0.5r\}$ in a Black-Scholes market with $r = 0.02$, $\alpha = 0.06$ and $\sigma = 0.2$ and investment horizon $T = 10$

$$\pi_t^* = -\frac{\alpha - r}{\sigma^2}(2k(t)k'(t) - 2k'(t) + (k^2(t) - 2k(t))\sigma^2(\pi^*(t))^2)\frac{a^2(t)}{c(t)}.$$

Now, using that $\left(\frac{\sigma^2}{\alpha-r}\pi^*(t) + 1\right)(2k(t) - k^2(t))^{-1} = \frac{a^2(t)}{c(t)}$ we get

$$\pi_t^* = k_0(t) + k_1(t)\pi^*(t) + k_2(\pi^*(t))^2 + k_3(\pi^*(t))^3, \tag{43}$$

with $k_1(t) = \frac{1-k(t)}{k(t)-\frac{1}{2}k^2(t)}k'(t)$, $k_0(t) = \frac{\alpha-r}{\sigma^2}k_1(t)$, $k_2 = \alpha - r$ and $k_3 = \sigma^2$. In the natural case where $k(T) = 1$, we have the boundary condition $\pi^*(T) = 0$. For the function $k(t) = e^{-\bar{r}(T-t)}$, we see that also $\pi_t^*(T) = 0$ and that $\bar{r} > 0$ is a sufficient condition for $k_1(t) > 0$ for $t < T$. As the t-derivative in Eq. (43), can be rewritten as $\pi_t^* = (k_1(t)/\sigma^2 + (\pi^*)^2)((\alpha - r) + \sigma^2\pi^*)$, it is concluded that $\pi_t^* > 0$ whenever $\pi^*(t) > -\frac{(\alpha-r)}{\sigma^2}$ for $t < T$. Thus, the strategy is negative and increasing towards 0 before termination. Also, the strategy must be increasing with lower bound for $\pi^*(t)$ given by $-\frac{(\alpha-r)}{\sigma^2}$. As a result, the polynomial in Eq. (43) is Lipschitz for $t \in [0, T]$ and thus the ODE has a unique solution (at least for $\bar{r} > 0$). Also, as we conclude that the strategy is negative for all $t < T$, we see that a sufficient condition for $c(t) > 0$ is $(\alpha - r)^2 \leq 2\sigma^2 r$, since we hereby achieve $c'(t) < 0$ with terminal condition $c(T) = 1$.

The strategy can be viewed in a numerical example in Fig. 3 for the function $k(t) = e^{-\bar{r}(T-t)}$ for some $\bar{r}$. We have included numerical results for $\bar{r} < 0$ although existence of a solution has not been shown. We observe that the strategies are all negative, strictly increasing and terminating in zero. This was expected for $\bar{r} > 0$. We note that for $\bar{r} = 0$ it is optimal to have everything in the bank account as this minimize the variance. For every other $\bar{r}$, it is optimal for the investor to short-sell stocks at all times. This is because the investor at all times has an objective where it is beneficial to have a small absolute expected wealth: (1) When $\bar{r} > 0$ the desire arises as the target $k(t)E_{t,x}[X^{\pi^*}(T)]$ becomes small, (2) When $\bar{r} < 0$, then the desire arises from the investor seeking to actively make the target smaller, such that it is easier to achieve.

In this case, we do have existence and uniqueness of the optimal control for some functions k and for others not. The strategy is specified as a time-dependent proportion invested in stocks. Such a specification is desirable from a practical point of view. However, for the special cases studied in detail here (exponential k) the proportion is negative and increasing which reduces its impact from a practical point of view.

4.2 Endogenous habit formation mean–variance style

In this subsection, we consider the optimization problem where the investor has the objective to maximize

$$(l_1(t)x + l_2(t))\, E_{t,x}\left[X^{\pi}(T)\right] - \frac{1}{2}E_{t,x}\left[(X^{\pi}(T) - k(t)x)^2\right],$$

for some functions l_1, l_2, k. The problem is an adjustment to the traditional mean–variance optimization, where we punish with the second order moment of the difference between terminal wealth and some projection of current wealth. To our knowledge, the result is new in its general form. With $l_1(t) = l_2(t) = 0$ the problem corresponds to the Endogenous Habit Formation problem originally solved by Kryger and Steffensen (2010).

The problem corresponds to the function f given by

$$\begin{aligned} f(t, x, y, z) &= (l_1(t)x + l_2(t))\, y - \frac{1}{2}\left(z + k^2(t)x^2 - 2k(t)xy\right), \\ & f_y l_1(t)x + l_2(t) + k(t)x, \\ & f_z = -\frac{1}{2}, \end{aligned}$$

and functions $g(x) = x$ and $h(x) = x^2$.

We search for a solution in the general form

$$G(t, x) = a(t)\,x + b(t),\;\; H(t, x) = c(t)x^2 + d(t)x + e(t),$$

with $a(T) = c(T) = 1$ and $b(T) = d(T) = e(T) = 0$. The partial derivatives are as before and thus, we see that the optimal investment candidate from Eq. (19) is in the general form in Eq. (20) with

$$p_1(t) = \frac{\alpha - r}{\sigma^2}\left(k(t)\frac{a(t)}{c(t)} + \frac{1}{2}\frac{a(t)}{c(t)}l_1(t) - 1\right),$$

$$p_2(t) = \frac{\alpha - r}{\sigma^2} \left(\frac{1}{2} \frac{a(t)}{c(t)} l_2(t) - \frac{1}{2} \frac{d(t)}{c(t)} \right),$$

where a, c, d are solutions to the system in Eqs. (21)–(25). The terminal value is specified through $\pi^*(T, x)x = \frac{\alpha - r}{\sigma^2} \left(\frac{1}{2} l_2(T) + [k(T) + \frac{1}{2} l_1(T) - 1]x \right)$. We note that for the strategy to realize the infimum in Eq. (13), we must have $-c(t) < 0$.

In this case, we do not have any existence and uniqueness results. The optimal strategy is affine in the wealth with mixed relations to practice since it adds an undesirable wealth-independent amount to the position from a desirable wealth-independent proportion.

4.2.1 Endogenous habit, mean–variance ($l_1(t) = 0$ and $l_2(t) = \frac{1}{\gamma}$)

The case with $l_1(t) = 0$ and $l_2(t) = \frac{1}{\gamma} > 0$ corresponds to the investor who seeks to maximize

$$E_{t,x} \left[X^\pi(T) \right] - \frac{\gamma}{2} E_{t,x} \left[\left(X^\pi(T) - k(t)x \right)^2 \right].$$

The problem is an alternative to traditional mean–variance optimization with constant γ. As a result, we get

$$p_1(t) = \frac{\alpha - r}{\sigma^2} \left(k(t) \frac{a(t)}{c(t)} - 1 \right),$$

$$p_2(t) = \frac{\alpha - r}{\sigma^2} \left(\frac{a(t)}{c(t)} \frac{1}{\gamma} - \frac{1}{2} \frac{d(t)}{c(t)} \right).$$

In conclusion, we are left with the full system of ODE's in Eqs. (21)–(25). We are not immediately able to verify the existence and uniqueness of the solutions to this system. We are however able to determine a solution numerically. We see that $\pi^*(T, x) = \frac{\alpha - r}{\sigma^2} (k(T) - 1)\, x + \frac{\alpha - r}{\sigma^2 \gamma}$. Again, for the resulting strategy to realize the infimum in Eq. (13), we must have $c(t) > 0$ for all t. This is satisfied in the following numerical case. In addition it is checked that $e(t) \geq 0$ and that $-2\sqrt{c(t)e(t)} \leq d(t) \leq 2\sqrt{c(t)e(t)}$, such that it is ensured that $H(t, x) \geq 0$ for all x.

Since the investment strategy is in the form $\pi^*(t, x) = p_1(t)x + p_2(t)$, we are not able to plot the strategy as a function of t. A plot of the functions p_1 and p_2 can be viewed in a numerical example in Fig. 4.

In the example we observe the following features: (1) If wealth is positive the optimal proportion is increasing in time, (2) The optimal proportion is decreasing in wealth, and at every point in time there exists a wealth threshold, such that the optimal proportion is negative if and only if wealth exceeds that threshold (which itself is increasing over time).

4.2.2 Endogenous habit, scaled mean–variance ($l_1(t) = \frac{1}{\gamma}$ and $l_2(t) = 0$)

In the case with $l_1(t) = \frac{1}{\gamma} > 0$ and $l_2(t) = 0$, the investor seeks to maximize

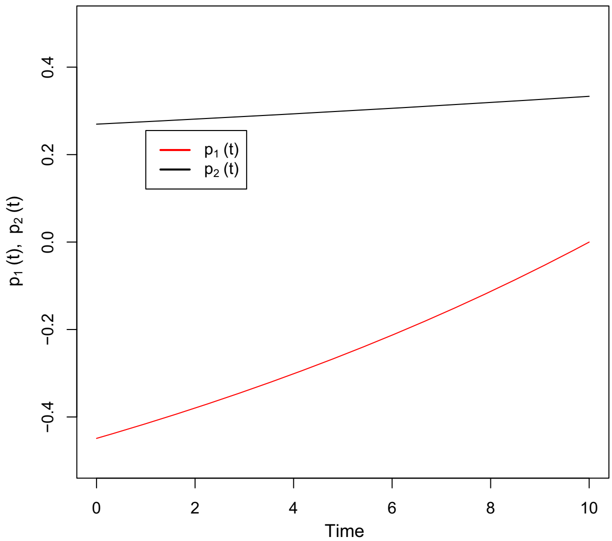

Fig. 4 Functions p_1 and p_2, defining the optimal amount invested in stocks by $\pi^*(t,x)x = p_1(t)x + p_2(t)$ for the investor with Endogenous habit formation mean–variance style with $\gamma = 3$ and $k(t) = e^{\bar{r}(T-t)}$ for $\bar{r} = 0.04$ in a Black-Scholes market with $r = 0.02$, $\alpha = 0.06$ and $\sigma = 0.2$ and investment horizon $T = 10$

$$E_{t,x}\left[X^{\pi}(T)\right] - \frac{\gamma}{2}\frac{E_{t,x}\left[(X^{\pi}(T) - k(t)x)^2\right]}{x}.$$

The problem is an adjustment to the scaled mean–variance optimization. We get

$$\begin{aligned} p_1(t) =& \frac{\alpha - r}{\sigma^2}\left(k(t)\frac{a(t)}{c(t)} + \frac{a(t)}{c(t)}\frac{1}{\gamma} - 1\right), \\ p_2(t) =& \frac{\alpha - r}{\sigma^2}\left(-\frac{1}{2}\frac{d(t)}{c(t)}\right), \end{aligned}$$

We see that the guess $d(t) = 0$ leads to

$$\pi^*(t) = p_1(t) = \frac{\alpha - r}{\sigma^2}\left(\left[k(t) + \frac{1}{\gamma}\right]\frac{a(t)}{c(t)} - 1\right). \tag{44}$$

In conclusion, $\pi^*(t)$ is independent of x and thus we are again in the case of Sect. 3.1.2 and the functions a and c solve the ODE system in Eqs. (29)–(30) with π^* specified in Eq. (44) above. In this case we must check that $-\frac{\gamma}{x}c(t) < 0$, for the strategy to realize the infimum in Eq. (13). This can be achieved if $\gamma > 0$ and $c(t) > 0$ for all t and if the wealth of the investor is positive at all times t. First, since the investment proportion is dependent on t only, the wealth process is a generalized GBM and in conclusion $X^{\pi^*}(t) > 0$ at all times. A sufficient condition for $c(t) \geq 0$ is again $(\alpha - r)^2 \leq 2\sigma^2 r$.

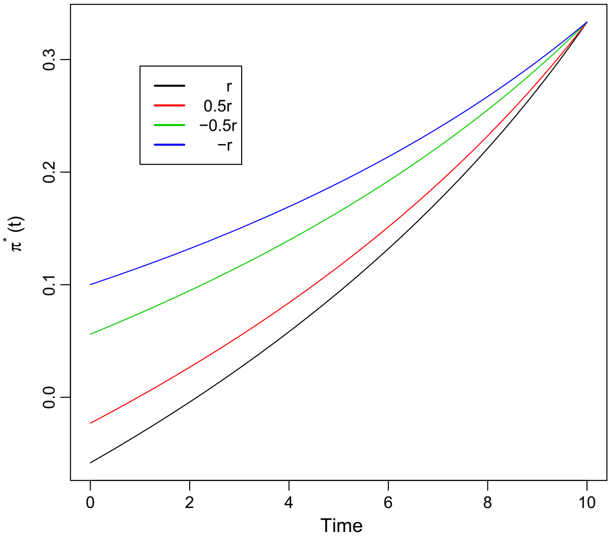

Fig. 5 Optimal proportion of wealth invested in stocks, $\pi^*(t)$, for the investor with Endogenous habit formation scaled mean–variance style with $\gamma = 3$ and $k(t) = e^{\bar{r}(T-t)}$ for $\bar{r} \in \{-r, -0.5r, 0.5r, r\}$ in a Black-Scholes market with $r = 0.02$, $\alpha = 0.06$ and $\sigma = 0.2$ and investment horizon $T = 10$

With this structure, we are again able to derive an ODE for π. As a result, we do not have to calculate a and c in order to derive π^*, but instead just solve

$$\pi_t^* = k_0(t) + k_1(t)\pi^*(t) + k_2\pi^*(t)^2 + k_3\pi^*(t)^3, \tag{45}$$

with $k_0(t) = (k'/(k+1/\gamma)+r)(\alpha-r)/\sigma^2$, $k_1(t) = k'/(k+1/\gamma)+r+\theta^2$, $k_2 = 2(\alpha - r)$, and $k_3 = \sigma^2$. The boundary condition is $\pi^*(T) = \frac{\alpha-r}{\sigma^2}\left(k(T) + \frac{1}{\gamma} - 1\right)$. For the natural choice of $k(T) = 1$, we thus have the boundary condition $\pi^*(T) = \frac{\alpha-r}{\sigma^2\gamma}$. For the choice $k(t) = e^{\bar{r}(T-t)}$, we observe that a sufficient condition for all the coefficients to be positive is $\bar{r} \leq r$. In this case we have ensured an increasing investment strategy when π^* is positive. Due to the t-dependence of the coefficients, and the possibility for π^* to be both positive and negative, it is out of scope to show existence and uniqueness of the solution to the ODE. However, for $\bar{r} = 0$ we are able to show that a sufficient condition for $\pi_t^* > 0$ for all values of π^* is $(\alpha - r)^2 \leq 4\sigma^2 r$. We notice that this is satisfied by the above-mentioned sufficient condition for c to be positive.

The strategy can be viewed in a numerical example in Fig. 5. We observe that the investment strategy is a strictly increasing proportion terminating in the Merton proportion. It is seen, that the investment strategy remains positive at all times when $\bar{r} > 0$. This corresponds to the cases where $\bar{r}$ denotes a target rate of return. When $\bar{r} = r$, then the penalty alone is minimized by full investment in the bank account, but since the optimization is also concerned with maximizing the mean, we arrive at a strictly positive investment strategy.

In this case, we do not have any existence and uniqueness results. The strategy is specified as a time-dependent proportion invested in stocks. For the special cases studied in detail here (exponential k) the proportion turns out to be increasing. However, similarly to Sect. 4.1.2, we can vary the time-dependence and the monotonicity by introducing time-dependence of γ.

4.2.3 Endogenous habit ($l_1(t) = l_2(t) = 0$)

The special case where $l_1(t) = l_2(t) = 0$ is relevant when investors have a time dependent return target, k and seeks to minimize the quadratic distance

$$\frac{1}{2} E_{t,x} \left[(X^{\pi}(T) - k(t)x)^2 \right],$$

This problem was first solved by Kryger and Steffensen (2010). We observe that

$$\begin{aligned} p_1(t) =& \frac{\alpha - r}{\sigma^2} \left(k(t) \frac{a(t)}{c(t)} - 1 \right), \\ p_2(t) =& - \frac{\alpha - r}{\sigma^2} \frac{1}{2} \frac{d(t)}{c(t)}. \end{aligned}$$

We see that the guess $d(t) = 0$ leads to

$$\pi^*(t) = p_1(t) = \frac{\alpha - r}{\sigma^2} \left(k(t) \frac{a(t)}{c(t)} - 1 \right). \tag{46}$$

In conclusion, $\pi^*(t)$ is independent of x and thus we are again in the case of Sect. 3.1.2 and the functions a and c solve the ODE system in Eqs. (29)–(30) with π^* specified in Eq. (46) above. We require that $c(t) > 0$ for all t for the strategy to realize the infimum in Eq. (13).

With this structure, we are able to derive the following ODE for π. As a result, we do not have to calculate a and c in order to derive π^*.

$$\pi_t^* = \frac{\alpha - r}{\sigma^2} \frac{c\,(k_t a + k a_t - c_t) - c_t\,(ka - c)}{c^2} \tag{47}$$

$$= k_0(t) + k_1(t)\pi^*(t) + k_2 \pi^*(t)^2 + k_3 \pi^*(t)^3, \tag{48}$$

with $k_0(t) = (k'/k + r)(\alpha - r)/\sigma^2$, $k_1(t) = k'/k + r + \theta^2$, $k_2 = 2(\alpha - r)$, and $k_3 = \sigma^2$. The boundary condition is $\pi^*(T) = \frac{\alpha - r}{\sigma^2}\,(k(T) - 1)$.

Because of the terms $\left(k'/k + r\right)(\alpha - r)/\sigma^2$ the solution is not zero, although $\pi^*(T) = 0$ for $k(T) = 1$, which is the more meaningful value for $k(T)$. The quantity $-k'/k$ represents the target rate of return of the investor. Therefore, it is reasonable to let $-k'/k$ be a constant larger than r. If $-k'/k = r$, then the optimal strategy *is* zero, precisely because this target can be obtained via a full allocation to the bond.

In the natural case with $k(T) = 1$ and $-k'/k > r$, it is immediately seen that $\pi_t^*(T) < 0$. In conclusion, the strategy is positive and decreasing towards zero right

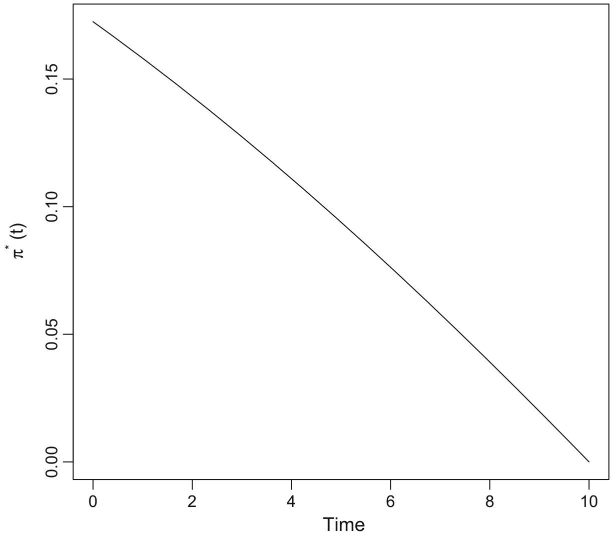

Fig. 6 Optimal proportion of wealth invested in stocks, $\pi^*(t)$, for the investor with endogenous habit formation with $k(t) = e^{\bar{r}(T-t)}$ for $\bar{r} = 0.04$ in a Black–Scholes market with $r = 0.02$, $\alpha = 0.06$ and $\sigma = 0.2$ and investment horizon $T = 10$

before time T. We may rewrite the differential equation for π_t^* to conclude that the only positive value for π^* for which $\pi_t^* = 0$ is $\frac{-\frac{\alpha-r}{\sigma}+\sqrt{(\frac{\alpha-r}{\sigma})^2-4(k'/k+r)}}{2\sigma}$. In conclusion, π_t^* will remain negative until this value is reached, and $\pi^*(t)$ will be a decreasing function (increasing backwards in time until reaching the maximum). Since $\pi^*(t)$ is thus bounded for $t \in [0, T]$, we may also conclude that the ODE expressed by the polynomial in Eq. (48) is Lipschitz and thus there exists a unique solution to the ODE.

The strategy can be viewed in a numerical example in Fig. 6 for $k(t) = e^{\bar{r}(T-t)}$ and $\bar{r} > r$. As expected, we observe that the strategy is a strictly decreasing proportion terminating in zero at time T. As mentioned above, for positive investments in stocks, it is required that the $\bar{r} > r$, such that it corresponds to a target rate of return higher than investment in the bank account.

In this case, we do have existence and uniqueness of the optimal control. The strategy is specified as a time-dependent proportion invested in stocks. This is desirable from a practical point of view. Further, it is even decreasing which is also in line with practical life-cycle investment advice, typically motivated in other ways. The termination at zero is, however, typically not part of this advice. Interestingly, such behavior also shows up in a very different theoretical framework, namely the so-called worst-case scenario portfolio optimization, introduced by Korn and Wilmott (2002) and developed further by many authors since then.

4.3 Generalized mean–standard-deviation

Inspired by the discussion in the preceding subsection it is natural to modify the problem, seemingly slightly, to penalize with standard deviation instead of variance.

In single-period models it is well-known that mean–variance and mean–standard-deviation are equivalent—in the sense that the set of risk aversions maps into the same set of controls. As it turns out, this equivalence does not carry over to the dynamic model.

The optimization problem considered in this subsection is that of an investor whose objective is to maximize

$$l_2(t)E_{t,x}\left[X(T)\right] - \sqrt{E_{t,x}\left[\left(X(T) - k(t)E_{t,x}\left[X(T)\right]\right)^2\right]}.$$

For some functions k and $\gamma > 0$. The problem has been solved in Kryger and Steffensen (2010) for $k(t) = 1$ and $l_2(t) = \frac{1}{\gamma}$, but is included in this paper for comparison with the other problems. We restrict ourselves to the case where γ is only dependent on time such that the currency unit match between the mean and the penalty term.

The problem corresponds to the function f given by

$$f(t,x,y,z) = l_2(t)y - \left(z - (2-k(t))k(t)y^2\right)^{\frac{1}{2}}, \tag{49}$$

$$f_y = l_2(t) + y(2-k(t))k(t)\left(z - (2-k(t))k(t)y^2\right)^{-\frac{1}{2}}, \tag{50}$$

$$f_z = -\frac{1}{2}\left(z - (2-k(t))k(t)y^2\right)^{-\frac{1}{2}}, \tag{51}$$

and functions $g(x) = x$ and $h(x) = x^2$.

We immediately search for a solution in the form

$$G(t,x) = a(t)x, \;\; H(t,x) = c(t)x^2,$$

with $c(t) \geq (2-k(t))k(t)a^2(t)$. The partial derivatives are as before, with $b(t) = d(t) = e(t) = 0$, such that the optimal investment candidate from Eq. (19) becomes

$$\pi^*(t) = \frac{\alpha - r}{\sigma^2}\left(\left[l_2(t)\left(\frac{c(t)}{a^2(t)} - (2-k(t))k(t)\right)^{\frac{1}{2}} + (2-k(t))k(t)\right]\frac{a^2(t)}{c(t)} - 1\right). \tag{52}$$

We see that the optimal strategy is independent of x and as a result we have the same structure of the solution as in Sect. 3.1.2 and we need to solve the system of ODE's specified in Eqs. (29)–(30) with $\pi^*(t)$ given in Eq. (52) above. We see that providing that $l_2(t) > 0$ and $c(t) \geq (2-k(t))k(t)a^2(t)$, then the strategy realizes the infimum in Eq. (13) when the wealth x is positive at all times. This is immediately achieved since the strategy $\pi^*(t)$ is only t-dependent and thus produces a generalized GBM wealth process.

4.3.1 Mean–standard-deviation ($k(t) = 1$)

In the special case where $k(t) = 1$, we have a traditional formulation of a mean–standard-deviation problem

$$l_2(t)E_{t,x}\left[X\left(T\right)\right] - \sqrt{Var_{t,x}\left[X\left(T\right)\right]}.$$

We get

$$\pi^*(t) = \frac{\alpha - r}{\sigma^2}\left(l_2(t)\frac{\left(c(t) - a^2(t)\right)^{\frac{1}{2}} a(t)}{c(t)} + \frac{a^2(t)}{c(t)} - 1\right). \tag{53}$$

Surprisingly, the solution to the system in Eq. (29)–(30) with $\pi^*(t)$ given in Eq. (53) above is $\pi^*(t) = 0$ via $c(t) = a^2(t)$. For this solution,

$$a_t = -ra(t),\ a\left(T\right) = 1,$$
$$c_t = -2rc(t),\ c\left(T\right) = 1,$$

such that

$$a(t) = e^{r(T-t)},$$
$$c(t) = e^{2r(T-t)}.$$

We note that the infimum condition is satisfied by $c(t) = a^2(t)$. This solution of course makes the case less interesting, although even this is an important insight. The result was also derived as special cases in Kryger and Steffensen (2010) and Kronborg and Steffensen (2015).

In this case, we do have existence and uniqueness of the optimal control. The strategy is not to buy stocks. Although this is obviously in conflict with practice, we do find that it is very important to understand that and why such a strategy occurs in the mean–standard deviation case. Intuitively, over infinitesimally small time intervals the order of the $\sqrt{dt}$-term stemming from the variance of capital gains dominates the order of the dt-term stemming from the expectation of capital gains. Working out the strategy backwards, one should never start investing in stocks in the first place. This intuition also explains why there is no function l that can prevent this conclusion.

4.4 Endogenous habit formation mean–standard-deviation style

In this subsection, we consider the optimization problem where the investor has the objective to maximize

$$l_2(t)E_{t,x}\left[X^{\pi}\left(T\right)\right] - \sqrt{E_{t,x}\left[(X^{\pi}\left(T\right) - k(t)x)^2\right]},$$

for some function γ. The problem is an adjustment to the traditional mean–standard-deviation optimization, where we punish with the second order moment of the difference between terminal wealth and some projection of current wealth. Again, we restrict ourselves to scaling the penalty term with a t-dependent function γ such that the currency units of the problem are consistent. To our knowledge, the result is new.

The problem corresponds to the function f given by

$$f(t,x,y,z) = l_2(t)y - \sqrt{z + k^2(t)x^2 - 2k(t)xy}, \tag{54}$$

$$f_y = l_2(t) + k(t)x\left(z + k^2(t)x^2 - 2k(t)xy\right)^{-\frac{1}{2}}, \tag{55}$$

$$f_z = -\frac{1}{2}\left(z + k^2(t)x^2 - 2k(t)xy\right)^{-\frac{1}{2}}, \tag{56}$$

and functions $g(x) = x$ and $h(x) = x^2$.

Again, we immediately search for a solution in the form

$$G(t,x) = a(t)x, \quad H(t,x) = c(t)x^2,$$

with $a(T) = c(T) = 1$ and the requirement $c(t) \geq (2k(t)a(t) - k^2(t))$, such that the square root in Eqs. (54)–(56) is meaningful. The partial derivatives are as before, with $b(t) = d(t) = e(t) = 0$, and thus, the optimal investment candidate from Eq. (19) becomes

$$\pi^*(t) = \frac{\alpha - r}{\sigma^2}\left(\frac{l_2(t)\left(c(t) + k^2(t) - 2k(t)a(t)\right)^{\frac{1}{2}} a(t)}{c(t)} + \frac{k(t)a(t)}{c(t)} - 1\right). \tag{57}$$

Since the strategy is independent of x, we are once again in the case of Sect. 3.1.2 and the functions a and c are determined by the system in Eqs. (29)–(30) with π^* specified by Eq. (57). For the strategy to realize the infimum in Eq. (13), we must have $c(t), l_2(t) > 0$ for all t in addition to the requirement $c(t) \geq (2k(t)a(t) - k^2(t))$. Again, a sufficient condition for $c(t) > 0$ is $(\alpha - r)^2 \leq \sigma^2 r$ when the strategy is negative. The other requirement is checked in the numerical case.

We are not immediately able to determine a single ODE for π^* but instead rely on solving the system for a and c and plugging into Eq. (57). The system is highly nonlinear [due to the specification of $\pi^*(t)$ in Eq. (57)]. We are able to determine a numerical approximation to the solution in R, with $k(t) = e^{-\bar{r}(T-t)}$ and $\bar{r} > 0$ and $l_2(t) = \frac{1}{3}$, it is however out of scope to show the existence and uniqueness of a true solution. We see that the strategy has terminal value

$$\pi^*(T) = \frac{\alpha - r}{\sigma^2}\left(l_2(T)\left(1 + k^2(T) - 2k(T)\right)^{\frac{1}{2}} + k(T) - 1\right).$$

A natural choice for k would result in $k(T) = 1$, giving terminal condition $\pi^*(T) = 0$.

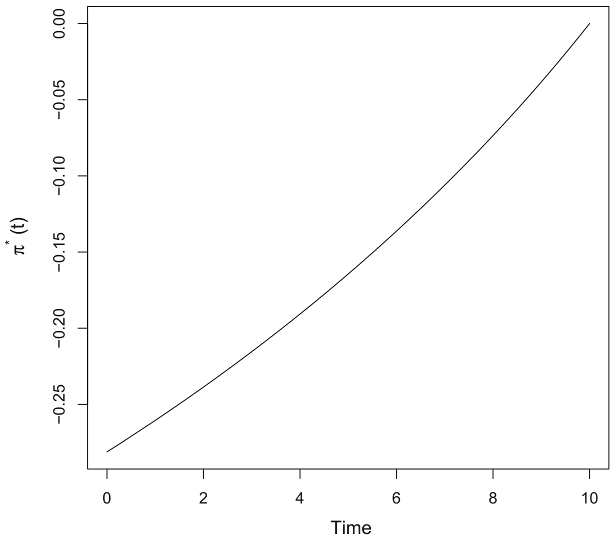

Fig. 7 Optimal proportion of wealth invested in stocks, $\pi^*(t)$, for the investor with Endogenous habit formation mean–standard-deviation style with $\gamma = 3$ and $k(t) = e^{-\bar{r}(T-t)}$ for $\bar{r} = 0.04$ in a Black–Scholes market with $r = 0.02$, $\alpha = 0.06$ and $\sigma = 0.2$ and investment horizon $T = 10$

The strategy can be viewed in a numerical example in Fig. 7 for $k(t) = e^{-\bar{r}(T-t)}$ with $\bar{r} = 0.04$ and constant $l_2(t) = \frac{1}{3}$. In this case the current wealth is *discounted* in the penalty term, leading to an investment strategy where it is optimal to short-sell stocks. We note that we are not able to find approximate solutions numerically in the more natural case with $\bar{r} < 0$. The numerical method seems to struggle with the square root and solutions might not exist e.g. due to the requirement $c(t) \geq (2k(t)a(t) - k^2(t))$. We are however able to guess the solution in the particular case of $\bar{r} = -r$. Here $c(t) = a^2(t) = k^2(t)$ is a solution, leading to the optimal investment strategy $\pi^*(t) = 0$.

In this case, we do not have any existence and uniqueness results. The strategy is specified as a time-dependent proportion invested in stocks. For the special cases studied here (exponential k) the proportion turns out to be negative and increasing which reduces its impact from a practical point of view. This compares to the case in Sect. 4.1.3 but from an analytical point of view it is noteworthy that the the proportion in Sect. 4.1.3 is concave whereas here it is convex.

4.5 Mean-scaled mean–variance optimization

In this subsection, we consider the optimization problem where the investor has the objective to maximize

$$l_2(t) E_{t,x}\left[X^\pi(T)\right] - \frac{1}{2}\frac{Var_{t,x}\left[X^\pi(T)\right]}{E_{t,x}\left[X^\pi(T)\right]},$$

for some function l_2. The problem is an adjustment to the traditional mean–variance optimization, such that the penalty term is in the same units as the mean. To our knowledge, the result is new. In Björk et al. (2014) a similar problem is solved by scaling with the current level of wealth x, instead of scaling with the expectation as above. We replicated this result as a special case of our mean–variance optimization in Sect. 4.1.2.

The above problem corresponds to the function f given by

$$
\begin{aligned}
f(t,x,y,z) &= l_2(t)y - \frac{1}{2}\frac{z-y^2}{y},\\
f_y &= \left(l_2(t) + \frac{1}{2}\right) + \frac{1}{2}\frac{z}{y^2},\\
f_z &= -\frac{1}{2}\frac{1}{y},
\end{aligned}
$$

and functions $g(x) = x$ and $h(x) = x^2$.

We again immediately search for a solution in the form

$$
G(t,x) = a(t)x, \quad H(t,x) = c(t)x^2,
$$

with $a(T) = c(T) = 1$. The partial derivatives are as before, with $b(t) = d(t) = e(t) = 0$ and thus, the optimal investment candidate in Eq. (19) becomes

$$
\pi^*(t) = \frac{\alpha - r}{\sigma^2}\left(\frac{1}{2}[2l_2(t)+1]\frac{a^2(t)}{c(t)} - \frac{1}{2}\right). \tag{58}
$$

We arrive at a $\pi^*(t)$ which is independent of x with $b(t) = d(t) = e(t) = 0$, and see that we are in the case of Sect. 3.1.2. As a result, the functions a and c are determined as the solutions to the system of ODE's given in Eqs. (29)–(30) with the optimal strategy specified by Eq. (58) above. We note that for the strategy to realize the infimum of Eq. (13), we must have $\frac{c(t)}{a(t)x} > 0$ for all (t,x). Since $\pi^*(t)$ is only dependent on t, the wealth process is positive as it is a generalized GBM and thus all we need to show is $\frac{c(t)}{a(t)} > 0$.

Again we instead consider the differential equation for π^* and see that

$$
\pi_t^* = \frac{\alpha - r}{\sigma^2}\left(l_2'(t) + \frac{1}{2}[2l_2(t)+1]\sigma^2\left(\pi^*(t)\right)^2\right)\frac{a^2(t)}{c(t)}.
$$

Since we have $\frac{a^2(t)}{c(t)} = \left(2\pi^*(t)\frac{\sigma^2}{\alpha - r} + 1\right)[2l_2(t)+1]^{-1}$, we get

$$
\pi_t^* = \frac{\alpha - r}{\sigma^2}\left(2l_2'(t)[2l_2(t)+1]^{-1} + \sigma^2\left(\pi^*(t)\right)^2\right)\left(\pi^*(t)\frac{\sigma^2}{\alpha - r} + \frac{1}{2}\right),
$$

with boundary condition $\pi^*(T) = \frac{\alpha - r}{\sigma^2} l_2(T)$. Thus, we are able to determine the optimal strategy by solving a single ODE and without determining the functions a and c.

We see that in the special case where $2l_2(t) := 1 - \frac{\tilde{c}(t)}{\tilde{a}^2(t)} + 2\frac{1}{\gamma}\frac{1}{\tilde{a}(t)}$, with $\tilde{a}$ and $\tilde{c}$ being the solution to the ODE's in Eqs. (40)–(41), the optimal strategy is identical to the mean–variance strategy with $l_1(t) = \frac{1}{\gamma}$ in Sect. 4.1.2.

4.5.1 Mean-scaled mean–variance ($l_2(t) = \frac{1}{\gamma}$)

In the natural case where $l_2(t) = \frac{1}{\gamma} > 0$, the investor wishes to maximize

$$E_{t,x}\left[X^\pi(T)\right] - \frac{\gamma}{2}\frac{Var_{t,x}\left[X^\pi(T)\right]}{E_{t,x}\left[X^\pi(T)\right]}.$$

The investment problem corresponds to the special case of the generalized mean–variance investor of Sect. 4.1.2, where instead of scaling the variance with current wealth, the variance is scaled with the expected terminal wealth. We consider this a more natural scale, since we are considering the variance of terminal wealth. As a result, we get the t-derivative of the optimal strategy

$$\pi_t^* = \sigma^2\left(\pi^*(t)\right)^3 + \frac{1}{2}(\alpha - r)\left(\pi^*(t)\right)^2, \tag{59}$$

with boundary condition $\pi^*(T) = \frac{\alpha - r}{\sigma^2\gamma}$. We observe the elegant structure of the solution which is captured in a single ODE for the optimal strategy $\pi^*(t)$ in Eq. (59). In comparison, the optimal strategy of the investor where the variance is scaled with current wealth, as expressed in Eq. (42) is still dependent of the a- and c-function and as a result, a system of two ODE's must be solved. We see, that $\pi_t^* \geq 0$ for $\pi^* \geq -\frac{1}{2}\frac{\alpha - r}{\sigma^2}$ and negative otherwise. In conclusion, we have a lower bound for $\pi^*(t)$ for $t \in [0, T]$ since the terminal condition is positive. Thus, we may conclude that the polynomial in Eq. (59) is Lipschitz and in conclusion, there exists a unique solution to the ODE. We approximate the solution numerically in R.

The strategy from Eq. (59) can be viewed in a numerical example in Fig. 8. We note that for the positive strategy to realize the infimum of Eq. (13), we must check that $\frac{c(t)}{a(t)} > 0$. This is satisfied in the numerical example. As expected, we see that the strategy is increasing and terminating in the Merton proportion.

In this case, we do have existence and uniqueness of the optimal control. The strategy is specified as an increasing time-dependent proportion invested in stocks, comparable with the case in Sect. 4.1.2. We refer to the discussion there about introducing time-dependence in the risk aversion to vary the monotonicity of the proportion. From comparing (42) and (59) we note carefully the relation between the solutions to the two problem: The terms of the derivative of π in (59) are also present in (42). In (59) existence and uniqueness thereafter can be concluded by Lipschitz arguments, whereas in (42) the situation is much more involved due to an extra term involving

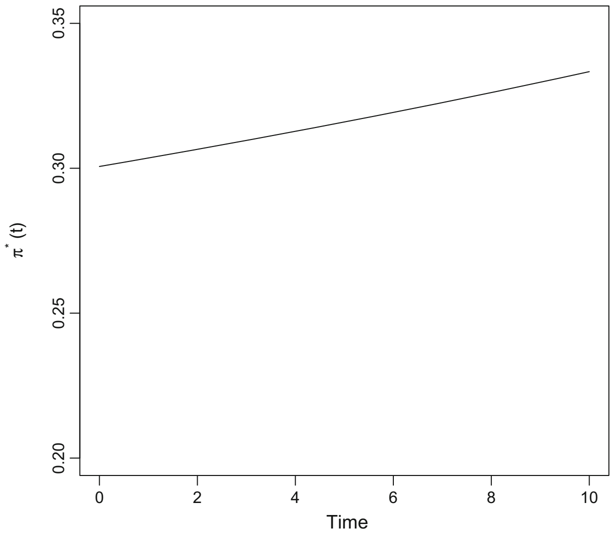

Fig. 8 Optimal proportion of wealth invested in stocks, $\pi^*(t)$, for the investor with mean-scaled mean–variance objective with $\gamma = 3$ in a Black–Scholes market with $r = 0.02$, $\alpha = 0.06$ and $\sigma = 0.2$ and investment horizon $T = 10$

other functions that are integrate parts of the solution. We find this to be an appealing observation and suggest to look more into scaling by expected wealth rather than current wealth in future studies on scaled variance objectives, i.e. replace

$$E_{t,x}\left[X^{\pi}(T)\right] - \frac{\gamma}{2}\frac{Var_{t,x}\left[X^{\pi}(T)\right]}{x}$$

by

$$E_{t,x}\left[X^{\pi}(T)\right] - \frac{\gamma}{2}\frac{Var_{t,x}\left[X^{\pi}(T)\right]}{E_{t,x}\left[X^{\pi}(T)\right]}.$$

We have presented a long list of problems and solutions in compendium-style, including some with little practical impact and/or with no existence and uniqueness results. It is then appropriate to conclude with this special case. The structure of the numerical results compares with other scaled variance studies in the literature, see Björk et al. (2014) and Björk et al. (2017), but the easier access to existence and uniqueness results is a genuine improvement from an analytical point of view.

Acknowledgements We would like to thank Alexander Lollike for the very helpful comments and considerations regarding the specification of the corrected verification theorem. In addition, we would like to thank the anonymous referee and the associate editor for useful comments and suggestions.

Appendix

Proof (Proof of Theorem 1) Consider an arbitrary admissible strategy π in the sense of Definition 2.

1. It is immediately observed, that for the arbitrary admissible strategy π, then the function $Y^{\pi}(t,x) \in C^{1,2}$, satisfying Eqs. (7)–(8) while having the process of Eq. (11) in $\mathcal{L}^2$, has the interpretation

$$Y^{\pi}(t,x) = y^{\pi}(t,x), \tag{60}$$

with y^{π} defined in Eq. (3).
This is a direct result of the Feynman–Kac stochastic representation formula (see e.g. Proposition 5.5 in Björk (2009)). The same applies for $z^{\pi}(t,x)$ with the function $Z^{\pi}(t,x) \in C^{1,2}$ specified in Definition 2.
As a direct consequence, it is concluded that the functions $G(t,x)$ and $H(t,x)$ have the interpretation $y^{\pi^*}(t,x)$ and $z^{\pi^*}(t,x)$.
2. Next, recalling that $J^{\pi}(t,x)$ denotes the value function of a given strategy π, we show that

$$\liminf_{h\to 0} \frac{J^{\pi^*}(t,x) - J^{\pi_h^*}(t,x)}{h} \geq 0,$$

for all (t,x), with π_h^* defined as in Definition 1 and π^* as specified in Eq. (13) of the theorem. First, we apply the rewriting of Remark 1. Second, re-calling that π^* is determined by Eq. (13), we conclude that for any fixed (t,x) and arbitrarily chosen π, the strategy π^* satisfies

$$\begin{aligned} & f_t - \left(r + \pi^*(\alpha - r)\right) x \left(f_y G_x + f_z H_x\right) - \frac{1}{2}\sigma^2(\pi^*)^2 x^2 \left(f_y G_{xx} + f_z H_{xx}\right) \\ & \quad \leq f_t - (r + \pi(\alpha - r))\, x \left(f_y G_x + f_z H_x\right) - \frac{1}{2}\sigma^2 \pi^2 x^2 \left(f_y G_{xx} + f_z H_{xx}\right), \end{aligned} \tag{61}$$

where we recall the suppressed arguments of the f-function, $f = f(t,x,G(t,x), H(t,x))$. Meanwhile, we observe that for any admissible strategy π, the value function $J^{\pi}(t,x) := f(t,x,y^{\pi}(t,x), z^{\pi}(t,x)) = f(t,x,Y^{\pi}(t,x), Z^{\pi}(t,x))$, by the chain-rule, has t-derivative

$$J_t^{\pi} = f_t^{\pi} + f_y^{\pi} Y_t^{\pi} + f_z^{\pi} Z_t^{\pi},$$

where we use the short notation $f^{\pi} = f(t,x,Y^{\pi}(t,x), Z^{\pi}(t,x))$, to highlight the arguments of the f-function. Since we are considering admissible strategies, we can use the interrelations in Eqs. (7) and (9), to arrive at

$$\begin{aligned} J_t^{\pi} &= f_t^{\pi} + f_y^{\pi}\left(-(r + \pi(\alpha - r))\, x Y_x^{\pi} - \frac{1}{2}\sigma^2 (\pi)^2 x^2 Y_{xx}^{\pi}\right) \\ &\quad + f_z^{\pi}\left(-(r + \pi(\alpha - r))\, x Z_x^{\pi} - \frac{1}{2}\sigma^2 (\pi)^2 x^2 Z_{xx}^{\pi}\right) \\ &= f_t^{\pi} - (r + \pi(\alpha - r))\, x \left(f_y^{\pi} Y_x^{\pi} + f_z^{\pi} Z_x^{\pi}\right) \end{aligned}$$

$$- \frac{1}{2}\sigma^2 (\pi)^2 x^2 \left(f_y^\pi Y_{xx}^\pi + f_z^\pi Z_{xx}^\pi \right).$$

Combined with the inequality of Eqs. (61), we see that

$$\begin{aligned} J_t^{\pi^*} \leq & f_t^{\pi^*} - \left(r + \pi_h^* (\alpha - r)\right) x \left(f_y^{\pi^*} G_x + f_z^{\pi^*} H_x \right) \\ & - \frac{1}{2}\sigma^2 (\pi^*)^2 x^2 \left(f_y^{\pi^*} G_{xx} + f_z^{\pi^*} H_{xx} \right), \end{aligned}$$

and we conclude that

$$\begin{aligned} J_t^{\pi_h^*} - J_t^{\pi^*} \geq\ & f_t^{\pi_h^*} - f_t^{\pi^*} \\ & - \left(r + \pi_h^* (\alpha - r)\right) x \left(f_y^{\pi_h^*} y_x^{\pi_h^*} + f_z^{\pi_h^*} z_x^{\pi_h^*} - f_y^{\pi^*} G_x - f_z^{\pi^*} H_x \right) \\ & - \frac{1}{2}\sigma^2 (\pi_h^*)^2 x^2 \left(f_y^{\pi_h^*} y_{xx}^{\pi_h^*} + f_z^{\pi_h^*} z_{xx}^{\pi_h^*} - f_y^{\pi^*} G_{xx} - f_z^{\pi^*} H_{xx} \right). \end{aligned}$$

Finally, we see that

$$Y^{\pi_h^*}(t, x) \to Y^{\pi^*}(t, x),$$

for $h \to 0$ (and similarly for Z). This is concluded as

$$\begin{aligned} Y^{\pi_h^*}(t, x) - Y^{\pi^*}(t, x) =& Y^{\pi^*}(t + h, x) - Y^{\pi^*}(t, x) \\ & - (Y^{\pi_h^*}(t + h, x) - Y^{\pi_h^*}(t, x)) \\ \simeq & Y_t^{\pi^*} \cdot h - Y_t^{\pi_h^*} \cdot h \to 0, \end{aligned}$$

for $h \to 0$, since the function $Y^{\pi^*}(t, s)$ are assumed to be in $C^{1,2}$ and by definition of π_h^*, we have $Y^{\pi^*}(t + h, x) = Y^{\pi_h^*}(t + h, x)$. Since it is also required that the function f is in $C^{1,2,2,2}$ in the sample space of the admissible strategies, we see that also

$$\begin{aligned} f_t^{\pi_h^*} &\to f_t^{\pi^*}, \\ f_y^{\pi_h^*} y_x^{\pi_h^*} &\to f_y^{\pi^*} G_x, \\ f_z^{\pi_h^*} z_x^{\pi_h^*} &\to f_z^{\pi^*} H_x, \\ f_y^{\pi_h^*} y_{xx}^{\pi_h^*} &\to f_y^{\pi^*} G_{xx}, \\ f_z^{\pi_h^*} z_{xx}^{\pi_h^*} &\to f_z^{\pi^*} H_{xx}, \end{aligned}$$

for $h \to 0$ and in conclusion we have shown that

$$\liminf_{h \to 0} \frac{J^{\pi^*}(t, x) - J^{\pi_h^*}(t, x)}{h} \geq 0.$$

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