

# Optimal consumption, investment and life insurance with surrender option guarantee

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## Abstract

We consider an investor, with an uncertain life time, endowed with deterministic labor income, who has the possibility to continuously invest in a Black-Scholes market and to buy life insurance or annuities. We solve the optimal consumption, investment and life insurance problem when the investor is restricted to fulfill an American capital guarantee. By allowing the guarantee to depend, in a very general way, on the past we include, among other possibilities, the interesting case of a minimum rate of return guarantee, commonly offered by pension companies. The optimal strategies turn out to be on option based portfolio insurance form, but since the capital guarantee is valid at every intermediate point in time, re-calibration is needed whenever the constraint is active.

*Keywords:* Stochastic control; martingale method; life insurance; rate guarantee; option based portfolio insurance; CRRA utility.

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# 1 Introduction

We unify two directions of generalizing the classical Merton's consumption-investment problem with labor income, namely the introduction of life insurance decisions and the introduction of capital constraints. This unification is particularly relevant since the savings in a life and pension contract often contain restrictions that are special examples of our capital constraint. We solve the joint problem where the individual with *an uncertain life time* can buy life insurance or annuities and, at the same time, there is a general *capital constraint* on his savings.

The *uncertain life time* is well-motivated but it turns a deterministic income rate *while being alive* into a stochastic income rate. Access to life insurance and annuities completes the market and makes the financial value of labor income unique.

The *capital constraint* allowed for in this paper is sufficiently general to cope with two well-motivated situations:

1) One possible capital constraint is that the savings are not allowed to become negative. We can speak of this as a no-borrowing constraint but it is important to stress that it is a constraint on wealth and not on investment position. In a life insurance context one would speak of such a constraint as a non-negative reserve constraint. The intuition is that it should always be the pension institution that owes money to the policy holder and not the other way around. This is a standard institutional/regulatory constraint that serves to separate, institutionally, saving business from lending business.

2) Another possible capital constraint is that the savings earn a minimum return. We speak of this as an interest rate guarantee. This is a delicate capital guarantee since it protects the dynamics of wealth rather than wealth itself. Thus, formalized as a constraint on the capital itself, this constraint depends on the trajectory of the state variables. Such a guarantee appears in many different types of life insurance contracts, ranging from participating life insurance with a so-called surrender option with a guaranteed sum upon surrender to unit-link life insurance where you can add on different types of options on the return.

In both situations one may ask why is the constraint there at all? The separation of saving business from lending business perhaps serves to protect the pension institution from unspanned policy holder credit risk. We stress the word *unspanned* here because one could argue that *spanned* life insurance risk is also credit risk in the context of borrowing against future income. Credit risk is a matter for the lending business who would require some material collateral, like a house or something, in addition to a 'planned' working life. But there is no such unspanned policy holder credit risk within our model, so who asked for a constraint? Also the guarantee on return may seem odd: Who asked for that? The titular question posed by Jensen and Sørensen (2001) means almost the same. Well, the answer 'the social planner' applies to both questions. If the society prefers that savings are non-negative, or if the society prefers that policy holders (via their delegated pension institutions) invest as safely as it is dictated by an interest rate guarantee, then the constraints can appear through the regulatory environment. Thus, these constraints should not necessarily be interpreted as a twist on the policy holders preferences within the model but as a societal constraint from outside the model/preferences. Formalistically, the two interpretations are the same, though. It is interesting to discuss if and when such constraints really are meaningful from the point of view of the social planner. However, this is beyond the scope of this paper. Furthermore, during these decades where certain markets show a tendency from classical interest rate guarantee participating life insurance towards modern unit-link products, the policy holders tend to actually buy these options on the return. So, could it be something that the people want, after all? Prospect theory may help motivating guarantees, see e.g. Døskeland and Nordahl (2008). We do not incorporate prospect theory in our preferences but take the guarantees to be exogenously given.

Life insurance was introduced to Merton's classical consumption-investment problem first already by Richard (1975) but Yaari (1965) had even earlier results in discrete time. It took three decades before the actuarial academic community realized the importance of the fundamental patterns of thinking by Richard (1975). Since then, many articles generalize the results by Richard in various directions. They include Pliska and Ye (2007) who allow for an unbounded

lifetime; Huang and Milevsky (2008) who allow for unspanned labor income; Huang et al. (2008) who separate the breadwinner income process from the family consumption process; Steffensen and Kraft (2008) who generalize to a multistate Markov chain framework typically used by actuaries for modeling a series of life history events; Nielsen and Steffensen (2008) who work with constraints on the insurance sum paid out upon death; Bruhn and Steffensen (2011) who generalize to a multiperson household, with focus on a married couple with economically and/or probabilistically dependent members; Kwak et al. (2011) who also consider a household but focus on generation issues.

Our contribution compared to the list of papers in the former paragraph is that we add capital constraints. They are particularly interesting since the life insurance and pension savings contracts typically contain borrowing or return constraints set by either the insurance company directly or by the regulatory authorities indirectly.

Capital constraints have been studied in a series of papers. They were first studied as a European (i.e. terminal) capital constraint in Teplá (2001) and Jensen and Sørensen (2001) who, interestingly, motivated the study of capital constraints from guarantees in pension contracts. Later results of El-Karoui et al. (2005) concern an American (i.e. continuous) capital guaranteed. The three papers on capital constraints mentioned above all deal with the investment problem exclusively, disregarding labor income and consumption. Kronborg (2011) generalizes the results of El-Karoui et al. (2005) to include spanned labor income and optimal consumption.

Our contribution compared to the list of papers in the former paragraph is that we add life insurance risk and decisions and allow for more general path dependent constraints. This combination makes it possible to study realistic decision processes and product designs including relevant regulatory and/or institutional constraints in the pension savings market.

Speaking of terminal and continuous constraints as being European and American, respectively, may seem a bit far-fetched until the resulting strategy appears: The best strategy is to follow a so-called option based portfolio insurance strategy (OBPI) where a certain part of the capital is invested in the optimal portfolio for the unconstrained problem and then put option protected downwards for the residual amount such that the guarantee is fulfilled and such that the combined position is worth the wealth. In the European case the portfolio is protected by a European put option whereas in the American case the portfolio is protected by an American put option. The underlying portfolio of the American put option is continuously updated, though, paid by the cash flow from the put option, see El-Karoui et al. (2005). A particular feature arises in the presence of income and consumption, namely, that also the strike is updated, see Kronborg (2011). This 'updating of the American strike' pattern appears also in the present paper in the presence of life insurance risk and protection. The update becomes even more delicate due to the path dependence of the capital guarantee.

Two other lines of literature should be mentioned for dealing with related problems although the market or the control fundamentally differ: Optimal annuitization has been studied in a series of papers, see Milevsky and Young (2007) and references therein. Optimal retirement timing has been studied in e.g. Farhia and Panageas (2007). We also wish to mention Dybvig and Liu (2010) since they combine the retirement decision with capital constraints similar to the ones we deal with here.

We apply the martingale method developed by Karatzas et al. (1987), Cox and Huang (1989), Cox and Huang (1991) and Cvitanic and Karatzas (1992). This method deals efficiently with capital constraints and has been used in all the papers on capital constraints mentioned above. However, Kraft and Steffensen (2012) show how also dynamic programming applies to problems with capital constraints.

The outline of the paper is as follows: In Section 2, we present the state processes and define the natural set of admissible strategies in presence of spanned labor income. In Section 3, we formalize and solve the unconstrained problem. Section 4 contains a formalization and a solution to the constrained problem. Numerical examples of the (constrained and unconstrained) optimal wealth dynamics are illustrated. We conclude in Section 5. Finally, a rigorous proof showing the admissibility of the strategy found in Section 4 is presented in the appendix.

## 2 Setup

Consider a policyholder with a life insurance policy issued at time 0 and terminated at a deterministic point in time  $0 < T < \infty$ . One should think of  $T$  as the time of retirement decided beforehand by the policyholder. The policyholder's time of death is given by a non-negative random variable  $\tau_d$  defined on a given probability space  $(\Omega, \mathcal{F}, P)$ . We assume that  $\tau_d$  has a probability distribution given by a function  $F$  with density function denoted by  $f$ . We can then define the *survivor function*  $\bar{F}$  by

$$\bar{F}(t) := P(\tau_d \geq t) = 1 - F(t) \text{ where } F(t) := P(\tau_d < t) = \int_0^t f(s)ds. \quad (1)$$

The instantaneous time- $t$  mortality rate of the policyholder is given by the hazard function

$$\mu(t) := \lim_{\epsilon \rightarrow 0} \frac{P(t \leq \tau_d < t + \epsilon \mid \tau_d \geq t)}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{F(t + \epsilon) - F(t)}{\epsilon} \frac{1}{\bar{F}(t)} = \frac{F'(t)}{\bar{F}(t)} = \frac{f(t)}{\bar{F}(t)} = -\frac{\partial}{\partial t} \log \bar{F}(t).$$

We get the following well-known expressions frequently used by actuaries

$$\bar{F}(t) = e^{-\int_0^t \mu(s)ds}, \quad (2)$$

$$f(t) = \mu(t)e^{-\int_0^t \mu(s)ds}. \quad (3)$$

In this paper we assume that the mortality rate is deterministic and given by a continuous function  $\mu : [0, \infty) \rightarrow [0, \infty)$ .

We now describe the investment and life insurance market available to the policyholder. Consider a Black-Scholes market consisting of a bank account,  $B$ , with risk free short rate,  $r$ , and a risky stock,  $S$ , with dynamics given by

$$\begin{aligned} dB(t) &= rB(t)dt, \quad B(0) = 1, \\ dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t), \quad S(0) = s_0 > 0. \end{aligned}$$

Here  $\alpha, \sigma, r > 0$  are constants and we assume  $\alpha > r$ . The process  $W$  is a standard Brownian motion on the probability space  $(\Omega, \mathcal{F}, P)$  equipped with the filtration  $\mathbb{F}^W = (\mathcal{F}^W(t))_{t \in [0, T]}$  given by the  $P$ -augmentation of the filtration  $\sigma\{W(s); 0 \leq s \leq t\}, \forall t \in [0, T]$ . We assume that the policyholder's random time of death  $\tau_d$  is stochastically independent of the filtration  $\mathbb{F}^W$ .

The policyholder is assumed to be endowed with a continuous deterministic labor income of rate  $\ell(t) \geq 0, \forall t \in [0, T]$ , as long as he is alive, and possibly an initial amount of money  $x_0 \geq 0$ . The policyholder should be thought of as an agent who has the possibility to invest and continuously consume from or save on a pension account. Naturally, we, as done by actuaries, refer to the policyholder's wealth, denoted by  $X$ , as the *reserve*. The reserve is maintained and invested by the pension company, but the policyholder chooses, in a continuous way, the risky asset allocation and the sum insured to be paid out upon death before time  $T$ <sup>1</sup>. More specifically, let  $c \geq 0$  be the rate of consumption and let  $\theta$  be the amount of money invested in the risky stock. Furthermore, let  $D \in [0, \infty)$  be the sum insured to be paid out upon death before time  $T$ . Choosing  $D$ , the policyholder agrees to hand over the amount of money  $X - D$  to the pension company upon death before time  $T$ , i.e. the pension company keeps the reserve  $X$  for themselves and pays out  $D$  as life insurance. Disregarding policy expenses the natural actuarial *risk premium* rate to pay for the life insurance  $D$  at time  $t$  is  $\mu(t)(D(t) - X(t))dt$ . Observe that choosing  $D > X$  corresponds to buying a life insurance and choosing  $D < X$  corresponds to selling a life insurance, i.e. buying an annuity. The dynamics of the reserve at the state alive<sup>2</sup>

<sup>1</sup>An alternative interpretation, also given by Nielsen and Steffensen (2008), is that the company chooses the investment and life insurance strategy on the behalf of the policyholder.

<sup>2</sup>For the rest of the paper we refer to *the reserve at the state alive* as simply *the reserve*.

becomes

$$dX(t) = [rX(t) + \theta(t)(\alpha - r) + \ell(t) - c(t) - \mu(t)(D(t) - X(t))]dt + \sigma\theta(t)dW(t), \quad t \in [0, T], \quad (4)$$

$$X(0) = x_0.$$

Once again we stress that in the case of death before the terminal time  $T$  only the sum insured,  $D$ , is paid out to the policyholder. We call the cash flow injected to the reserve given by  $\ell(t) - c(t) - \mu(t)D(t)$ ,  $\forall t \in [0, T]$ , the *pension contributions*.

We assume the trading strategy,  $\theta$ , to be an  $\mathcal{F}_t^W$ -adapted process with  $\int_0^T (\theta(t))^2 dt < \infty$  a.s., the consumption rate,  $c$ , to be a non-negative  $\mathcal{F}_t^W$ -adapted process with  $\int_0^T c(t) dt < \infty$  a.s., and the insurance strategy,  $D$ , to be a non-negative  $\mathcal{F}_t^W$ -adapted process with  $\int_0^T D(t) dt < \infty$  a.s. Note that this ensures that  $X$  is well-defined and we get the unique solution,  $\forall t \in [0, T]$ , known as the time- $t$  *retrospective reserve*

$$\begin{aligned} X(t) &= x_0 e^{\int_0^t (r+\mu(y))dy} + \int_0^t e^{\int_s^t (r+\mu(y))dy} [\theta(s)(\alpha - r) + \ell(s) - c(s) - \mu(s)D(s)] ds \\ &\quad + \int_0^t e^{\int_s^t (r+\mu(y))dy} \sigma\theta(s) dW(s). \end{aligned} \quad (5)$$

It is well known that in the Black-Scholes market the equivalent martingale measure,  $Q$ , is unique and given by the Radon-Nikodym derivative

$$\frac{dQ(t)}{dP(t)} = \Lambda(t) := \exp \left( - \int_0^t \frac{\alpha - r}{\sigma} dW(s) - \frac{1}{2} \int_0^t \left( \frac{\alpha - r}{\sigma} \right)^2 ds \right), \quad t \in [0, T], \quad (6)$$

and that the process  $W^Q$  given by

$$W^Q(t) = W(t) + \frac{\alpha - r}{\sigma} t, \quad t \in [0, T], \quad (7)$$

is a standard Brownian motion under the martingale measure  $Q$ . We get that the time- $t$  retrospective reserve can be represented,  $\forall t \in [0, T]$ , by

$$\begin{aligned} X(t) &= x_0 e^{\int_0^t (r+\mu(y))dy} + \int_0^t e^{\int_s^t (r+\mu(y))dy} [\ell(s) - c(s) - \mu(s)D(s)] ds \\ &\quad + \int_0^t e^{\int_s^t (r+\mu(y))dy} \sigma\theta(s) dW^Q(s). \end{aligned} \quad (8)$$

**Definition 2.1.** Define the set of admissible strategies, denoted by  $\mathcal{A}$ , as the consumption, investment and life insurance strategies for which the corresponding wealth process given by (8) is well-defined,

$$X(t) + g(t) \geq 0, \quad \forall t \in [0, T], \quad (9)$$

where  $g$  is the time- $t$  actuarial value of future labor income defined by

$$g(t) := \int_t^T e^{-\int_t^s (r+\mu(y))dy} \ell(s) ds,$$

and

$$E^Q \left[ \int_0^T e^{-\int_0^s (r+\mu(y))dy} \sigma\theta(s) dW^Q(s) \right] = 0. \quad (10)$$

The technical condition (10) is equivalent to the condition that under  $Q$  the process

$$\int_0^t e^{-\int_0^s (r+\mu(y))dy} \sigma \theta(s) dW^Q(s), \quad t \in [0, T],$$

is a martingale (in general it is only a local martingale, and also a supermartingale if (9) is fulfilled, see e.g. Karatzas and Shreve (1998)). From this we conclude that (10) insures that  $(c, \theta, D) \in \mathcal{A}$  if and only if  $X(T) \geq 0$  and,  $\forall t \in [0, T]$ ,

$$X(t) + g(t) = E^Q \left[ \int_t^T e^{-\int_t^s (r+\mu(y))dy} (c(s) + \mu(s)D(s)) ds + e^{-\int_t^T (r+\mu(y))dy} X(T) \middle| \mathcal{F}^W(t) \right]. \quad (11)$$

At time zero this means that the strategies have to fulfill the *budget constraint*

$$x_0 + g(0) = E^Q \left[ \int_0^T e^{-\int_0^t (r+\mu(y))ds} (c(t) + \mu(t)D(t)) dt + e^{-\int_0^T r ds} X(T) \right]. \quad (12)$$

For latter use we state the following remark.

**Remark 2.1.** Define

$$Z(t) := - \int_0^t e^{-\int_0^s (r+\mu(y))dy} (\ell(s) - c(s) - \mu(s)D(s)) ds + e^{-\int_0^t (r+\mu(y))dy} X(t), \quad t \in [0, T]. \quad (13)$$

By (8) we have that condition (10) is fulfilled if and only if  $Z$  is a martingale under  $Q$ . The natural interpretation is that under  $Q$  the discounted reserve minus the discounted pension contributions should be a martingale. It is useful to note that if  $Z$  is a martingale under  $Q$  the dynamics of  $X$  can be represented in the form

$$dX(t) = [(r + \mu(t))X(t) + \ell(t) - c(t) - \mu(t)D(t)]dt + \phi(t)dW^Q(t), \quad t \in (0, T], \quad (14)$$

for some  $\mathcal{F}^W(t)$ -adapted process  $\phi$ . Moreover, if the dynamics of  $X$  can be represented in the form given by (14) for some  $\mathcal{F}^W(t)$ -adapted process  $\phi$  satisfying  $\phi(t) \in \mathcal{L}^2$ ,  $\forall t \in [0, T]$ , then  $Z$  is a martingale under  $Q$ .

Finally, by condition (9) we allow the reserve process to become negative, as long as it does not exceed (in absolute value) the actuarial value of future labor income. Note that condition (9) puts a lower boundary on the reserve process and therefore rules out doubling strategies (see e.g. Karatzas and Shreve (1998)).

### 3 The unrestricted pension problem

For the unrestricted pension problem the policyholder chooses his consumption, investment and life insurance strategy in order to optimize the expected utility from consumption, bequest upon death, and terminal pension. That is, he searches for the strategy fulfilling the supremum

$$\sup_{(c, \theta, D) \in \mathcal{A}'} E \left[ \int_0^{T \wedge \tau_d} e^{-\int_0^t \beta(s) ds} u(c(t)) dt + K_1 e^{-\int_0^{\tau_d} \beta(s) ds} u(D(\tau_d)) \mathbf{1}_{(\tau_d < T)} + K_2 e^{-\int_0^T \beta(s) ds} u(X(T)) \mathbf{1}_{(\tau_d \geq T)} \right], \quad (15)$$

where  $u : \mathbb{R} \rightarrow [-\infty, \infty)$  is a utility function representing the policyholder's risk aversion and  $\beta$  is a deterministic function representing the policyholder's time preferences. The set  $\mathcal{A}'$ , called the set of *feasible* strategies, is a subset of the admissible strategies, defined by

$$\mathcal{A}' := \left\{ (c, \theta, D) \in \mathcal{A} \left| E \left[ \int_0^{T \wedge \tau_d} e^{-\int_0^t \beta(s) ds} \min(0, u(c(t))) dt + e^{-\int_0^{\tau_d} \beta(s) ds} K_1 \min(0, u(D(\tau_d))) 1_{(\tau_d < T)} + e^{-\int_0^T \beta(s) ds} K_2 \min(0, u(X(T))) 1_{(\tau_d \geq T)} \right] > -\infty \right. \right\}. \quad (16)$$

We see that it is allowed to draw an infinite utility from a strategy  $(c, \theta, D) \in \mathcal{A}'$ , but only if the expectation over the negative parts of the utility function is finite. Clearly, for a positive utility function we have that  $\mathcal{A}$  and  $\mathcal{A}'$  coincide. The constant weight factors  $K_1 > 0$  and  $K_2 > 0$  are measures of the policyholder's preferences concerning life insurance versus immediate consumption, and terminal pension versus immediate consumption, respectively. The case  $K_1 = 0$  has the well-known solution given by the pioneering work of Merton (1971). The case  $K_2 = 0$  and the case excluding consumption can be analyzed along the same lines as the analysis given in this paper. In order not to complicate the analysis further this will not be done since it requires special treatment. The results are given by natural modifications of the results obtained in this paper.

We want to write the optimization problem (15) in terms of the policyholder's instantaneous mortality rate. By the assumption that the random time of death,  $\tau_d$ , is stochastically independent of the filtration  $\mathbb{F}^W$ , and by use of (1), we can rewrite the expectation in (15) as

$$\begin{aligned} & E \left[ 1_{\{\tau_d \geq T\}} \int_0^T e^{-\int_0^t \beta(s) ds} u(c(t)) dt + 1_{\{\tau_d < T\}} \int_0^{\tau_d} e^{-\int_0^t \beta(s) ds} u(c(t)) dt \right. \\ & \quad \left. + K_1 e^{-\int_0^{\tau_d} \beta(s) ds} u(D(\tau_d)) 1_{(\tau_d < T)} + K_2 e^{-\int_0^T \beta(s) ds} u(X(T)) 1_{(\tau_d \geq T)} \right] \\ &= E \left[ \bar{F}(T) \int_0^T e^{-\int_0^t \beta(s) ds} u(c(t)) dt + \int_0^T f(y) \int_0^y e^{-\int_0^t \beta(s) ds} u(c(t)) dt dy \right. \\ & \quad \left. + K_1 \int_0^T e^{-\int_0^t \beta(s) ds} u(D(t)) f(t) dt + K_2 e^{-\int_0^T \beta(s) ds} u(X(T)) \bar{F}(T) \right]. \quad (17) \end{aligned}$$

Observe that

$$\begin{aligned} \int_0^T f(y) \int_0^y e^{-\int_0^t \beta(s) ds} u(c(t)) dt dy &= \int_0^T e^{-\int_0^t \beta(s) ds} u(c(t)) \int_t^T f(y) dy dt \\ &= \int_0^T e^{-\int_0^t \beta(s) ds} u(c(t)) [\bar{F}(t) - \bar{F}(T)] dt. \end{aligned}$$

Plug this into (17) to obtain

$$E \left[ \int_0^T e^{-\int_0^t \beta(s) ds} u(c(t)) \bar{F}(t) dt + K_1 \int_0^T e^{-\int_0^t \beta(s) ds} u(D(t)) f(t) dt + K_2 e^{-\int_0^T \beta(s) ds} u(X(T)) \bar{F}(T) \right]. \quad (18)$$

Finally, inserting (2) and (3) into (18) we can write the optimization problem (15) as

$$\sup_{(c, \theta, D) \in \mathcal{A}'} E \left[ \int_0^T e^{-\int_0^t (\beta(s) + \mu(s)) ds} [u(c(t)) + K_1 \mu(t) u(D(t))] dt + K_2 e^{-\int_0^T (\beta(s) + \mu(s)) ds} u(X(T)) \right]. \quad (19)$$

For the rest of this paper we assume that the policyholder's preferences towards risk are given by a constant relative risk aversion function (CRRA) in the form

$$u(x) = \begin{cases} \frac{x^\gamma}{\gamma}, & \text{if } x > 0, \\ \lim_{x \searrow 0} \frac{x^\gamma}{\gamma}, & \text{if } x = 0, \\ -\infty, & \text{if } x < 0, \end{cases}$$

for some  $\gamma \in (-\infty, 1) \setminus \{0\}$ .

**Remark 3.1.** We conjecture that the analysis presented in this paper can be extended to the case of a general utility function. This would, following the ideas of El-Karoui and Karatzas (1995) and El-Karoui et al. (2005), require use of the Gittins index methodology. We specialize to the case of CRRA utility due to the well-known property that the maximization problem (15) becomes linear in initial total wealth. Even in the case of a general path dependent constraint on the reserve (see Section 4) this feature allows us to derive fairly explicit expressions for the optimal strategies, leaving us with a greater understanding of the impact on the strategies caused by the introduced constraints.

### 3.1 How to choose $K_2$ - consumption after retirement

Choosing  $K_1$  is really a matter of personal preferences. How much do you love yourself relatively to your inheritors? However, choosing  $K_2$  can be done in a natural way. A common choice in the literature of consumption/portfolio selection is  $K_2 = 1$  by simply not allowing any weight constant. This corresponds to the case where the agent prefers a terminal wealth approximately equal to the size of accumulated consumption over the last year before the terminal time  $T$ . At least in a pension context this seems to be a very odd choice. What the policyholder (probably) wants is a terminal pension big enough for him to be able to maintain his standard of living throughout his uncertain remaining life time. At the same time, he should still take his time preferences into account. Define therefore the actuarial time- $T$  value of a life annuity paying 1 per year continuously until the time of death as

$$\bar{a}(T) := E \left[ \int_T^\infty e^{-\int_T^s r \, dy} \mathbf{1}_{(\tau_d \geq s)} ds \mid \tau_d > T \right] = \int_T^\infty e^{-\int_T^s (r + \mu(y)) dy} ds.$$

Disregarding policy expenses, the policyholder standing at the time of retirement,  $T$ , might use his terminal pension  $X(T)$  to buy, in the life insurance market, a life annuity of rate  $X(T)/\bar{a}(T)$ . Assume that he chooses to do so and thereafter simply consumes the entire life annuity (leaving no room for saving)<sup>3</sup>. We get that

$$\begin{aligned} & E \left[ \int_T^\infty e^{-\int_0^s \beta(y) dy} u \left( \frac{X(T)}{\bar{a}(T)} \right) \mathbf{1}_{(\tau_d \geq s)} ds \right] \\ &= E \left[ \int_T^\infty e^{-\int_0^s (\beta(y) + \mu(y)) dy} u \left( \frac{X(T)}{\bar{a}(T)} \right) ds \right] \\ &= E \left[ e^{-\int_0^T (\beta(y) + \mu(y)) dy} \int_T^\infty e^{-\int_T^s (\beta(y) + \mu(y)) dy} ds u \left( \frac{X(T)}{\bar{a}(T)} \right) \right] \\ &= E \left[ e^{-\int_0^T (\beta(y) + \mu(y)) dy} \bar{a}^S(T) u \left( \frac{X(T)}{\bar{a}(T)} \right) \right], \end{aligned} \tag{20}$$

where

$$\bar{a}^S(T) := E \left[ \int_T^\infty e^{-\int_T^s \beta(y) dy} \mathbf{1}_{(\tau_d \geq s)} ds \mid \tau_d > T \right] = \int_T^\infty e^{-\int_T^s (\beta(y) + \mu(y)) dy} ds$$

<sup>3</sup>The classical annuity result of Yaari (1965) proves that consuming the entire annuity income is optimal in the absence of bequest.



defines the subjective actuarial time- $T$  value of a life annuity paying 1 per year continuously until the time of death. Comparing the terminal term in (19) with (20) we get the natural choice

$$K_2 = \bar{a}^S(T)\bar{a}(T)^{-\gamma}. \quad (21)$$

### 3.2 Solving the unrestricted problem

We now solve the unrestricted optimization problem given by (15). In the calculations we focus on the representation of the problem given by (19). The methodology used in the following, introduced by Karatzas et al. (1987), Cox and Huang (1989), Cox and Huang (1991) and Cvitanic and Karatzas (1992), is known as the martingale method. One could, as done in Richard (1975), Pliska and Ye (2007), Nielsen and Steffensen (2008), Huang et al. (2008) and others, have used the Hamilton-Jacobi-Bellman (HJB) technique to derive the optimal consumption, investment and life insurance strategy. We choose the martingale approach since the solution to the restricted capital guarantee problem introduced in Section 4 is based on terms derived from the martingale method in the unrestricted case. Since the HJB technique seems to be the approach applied in the literature to solve such unrestricted optimization problems as (15) this subsection may also serve, in its own right, as an example of how to apply the martingale approach to a classic continuous time unrestricted optimization problem including life insurance. First we state the result in Proposition 3.1. The result is also obtained, in slightly different forms due to different setups and notation, by e.g. Richard (1975), Pliska and Ye (2007) and Nielsen and Steffensen (2008).

**Proposition 3.1.** *The optimal strategy for the problem (15) is given by the feedback forms*

$$c^*(t) := \frac{X(t) + g(t)}{f(t)}, \quad (22)$$

$$D^*(t) := \frac{X(t) + g(t)}{f(t)} K_1^{\frac{1}{1-\gamma}}, \quad (23)$$

$$\theta^*(t) := \frac{1}{1-\gamma} \frac{\alpha - r}{\sigma^2} (X(t) + g(t)), \quad (24)$$

where

$$g(t) := \int_t^T e^{-\int_t^s (r+\mu(y))dy} \ell(s) ds, \quad (25)$$

$$f(t) := \int_t^T e^{-\int_t^s (\tilde{r}(y)+\mu(y))dy} \left(1 + \mu(s) K_1^{\frac{1}{1-\gamma}}\right) ds + K_2^{\frac{1}{1-\gamma}} e^{-\int_t^T (\tilde{r}(y)+\mu(y))dy}, \quad (26)$$

and

$$\tilde{r}(t) := -\frac{\gamma}{1-\gamma} r - \frac{1}{2} \frac{\gamma}{(1-\gamma)^2} \left(\frac{\alpha - r}{\sigma}\right)^2 + \frac{1}{1-\gamma} \beta(t). \quad (27)$$

*Proof.* Let  $I : (0, \infty] \rightarrow [0, \infty)$  denote the inverse of the derivative of the utility function  $u$  and observe that by use of the concavity of  $u$  we have

$$u(x) \leq u(I(z)) - z(I(z) - x), \quad \forall x \geq 0, z > 0. \quad (28)$$

Define the adjusted state price deflator  $H$  as

$$H(t) := \Lambda(t) e^{\int_0^t (\beta(s) - r) ds},$$

and define  $\xi^* > 0$  as the constant satisfying

$$\begin{aligned} \mathcal{H}(\xi^*) := E^Q \left[ \int_0^T e^{-\int_0^t (r+\mu(s))ds} [I(\xi^* H(t)) + \mu(t) I(K_1^{-1} \xi^* H(t))] dt \right. \\ \left. + e^{-\int_0^T (r+\mu(s))ds} I(K_2^{-1} \xi^* H(T)) \right] = x_0 + g(0). \end{aligned} \quad (29)$$

Such a  $\xi^*$  clearly exist since  $I$  is continuous and decreasing and maps  $(0, \infty]$  onto  $[0, \infty)$  (see e.g. Karatzas and Shreve (1998), Chapter 3, Lemma 6.2).

Now, take an arbitrary strategy  $(c, \theta, D) \in \mathcal{A}'$  with corresponding wealth process  $(X(t))_{t \in [0, T]}$  as given. Using (28), the budget constraint (12) and finally (29) we get that

$$\begin{aligned} & E \left[ \int_0^T e^{-\int_0^t (\beta(s) + \mu(s)) ds} [u(c(t)) + K_1 \mu(t) u(D(t))] dt + K_2 e^{-\int_0^T (\beta(s) + \mu(s)) ds} u(X(T)) \right] \\ & \leq E \left[ \int_0^T e^{-\int_0^t (\beta(s) + \mu(s)) ds} [u(I(\xi^* H(t))) + K_1 \mu(t) u(I(K_1^{-1} \xi^* H(t)))] dt \right. \\ & \quad \left. + K_2 e^{-\int_0^T (\beta(s) + \mu(s)) ds} u(I(K_2^{-1} \xi^* H(T))) \right]. \end{aligned}$$

Since  $(c, \theta, D)$  was an arbitrarily chosen feasible strategy we obtain the *candidate* optimal strategy  $(c^*, \theta^*, D^*)$  given by

$$c^*(t) = I(\xi^* H(t)), \quad (30)$$

$$D^*(t) = I(K_1^{-1} \xi^* H(t)), \quad (31)$$

$$X^*(T) = I(K_2^{-1} \xi^* H(T)). \quad (32)$$

We have expressed the candidate optimal investment strategy in terms of the optimal terminal pension given by (32). Since  $(c^*, \theta^*, D^*)$ , by the definition of  $\xi^*$ , fulfills the budget constraint (12) it is well known that, in a complete market, there exist an investment strategy  $\theta^*$  such that  $X(T) = X^*(T)$  and  $(c^*, \theta^*, D^*)$  is admissible (see e.g. Cvitanic and Karatzas (1992) or Karatzas and Shreve (1998)). We are now going to calculate the hedging investment strategy  $\theta^*$  and to write the expressions (30) and (31) in more explicit forms.

Observing that  $I(x) = x^{-1/(1-\gamma)}$  we get from (29) that

$$\begin{aligned} \mathcal{H}(\xi^*) &= E \left[ \int_0^T H(t) e^{-\int_0^t (\mu(s) + \beta(s)) ds} (\xi^*)^{\frac{-1}{1-\gamma}} H(t)^{\frac{-1}{1-\gamma}} \left( 1 + \mu(t) K_1^{\frac{1}{1-\gamma}} \right) dt \right. \\ & \quad \left. + H(T) e^{-\int_0^T (\mu(s) + \beta(s)) ds} K_2^{\frac{1}{1-\gamma}} (\xi^*)^{\frac{-1}{1-\gamma}} H(T)^{\frac{-1}{1-\gamma}} \right] \\ &= (\xi^*)^{\frac{-1}{1-\gamma}} f(0), \end{aligned}$$

where we have defined

$$\begin{aligned} f(t) &= E \left[ \int_t^T e^{-\int_t^s (\mu(y) + \beta(y)) dy} \left( \frac{H(s)}{H(t)} \right)^{\frac{-\gamma}{1-\gamma}} \left( 1 + \mu(s) K_1^{\frac{1}{1-\gamma}} \right) ds \right. \\ & \quad \left. + e^{-\int_t^T (\mu(y) + \beta(y)) dy} K_2^{\frac{1}{1-\gamma}} \left( \frac{H(T)}{H(t)} \right)^{\frac{-\gamma}{1-\gamma}} \Big| \mathcal{F}^W(t) \right]. \quad (33) \end{aligned}$$

Since  $\mathcal{H}(\xi^*) = x_0 + g(0)$  per definition, we get that

$$\xi^* = (x_0 + g(0))^{-(1-\gamma)} f(0)^{1-\gamma}.$$

Inserting this into (30)–(32) and using the budget constraint (12) we get the following expressions for the candidate optimal strategy

$$\begin{aligned} c^*(t) &= \frac{X(t) + g(t)}{f(t)}, \\ D^*(t) &= \frac{X(t) + g(t)}{f(t)} K_1^{\frac{1}{1-\gamma}}, \\ X^*(T) &= \frac{X(t) + g(t)}{f(t)} \left( \frac{H(T)}{H(t)} \right)^{\frac{-1}{1-\gamma}} K_2^{\frac{1}{1-\gamma}}, \end{aligned}$$

and we recognize (22) and (23) from Proposition 3.1. To calculate  $f$  note that

$$E \left[ e^{-\int_t^s (\mu(y) + \beta(y)) dy} \left( \frac{H(s)}{H(t)} \right)^{\frac{-\gamma}{1-\gamma}} \middle| \mathcal{F}^W(t) \right] = e^{-\int_t^s \left( \mu(y) + \frac{1}{1-\gamma} \beta(y) - \frac{\gamma}{1-\gamma} r - \frac{1}{2} \frac{\gamma}{(1-\gamma)^2} \left( \frac{\alpha-r}{\sigma} \right)^2 \right) dy}. \quad (34)$$

Now combine (33) and (34) to obtain

$$f(t) = \int_t^T e^{-\int_t^s [\tilde{r}(y) + \mu(y)] dy} \left( 1 + \mu(s) K_1^{\frac{1}{1-\gamma}} \right) ds + K_2^{\frac{1}{1-\gamma}} e^{-\int_t^T [\tilde{r}(y) + \mu(y)] dy},$$

with  $\tilde{r}$  given by (27), which we identify as (26) from Proposition 3.1.

From the Girsanov Theorem we have

$$dH(t) = H(t) \left( (\beta(t) - r) dt - \frac{\alpha - r}{\sigma} dW(t) \right).$$

An application of Itô's lemma gives that

$$dX^*(t) = \dots dt + (X^*(t) + g(t)) \frac{1}{1-\gamma} \frac{\alpha - r}{\sigma} dW(t). \quad (35)$$

If we compare (35) with the  $X$ -dynamics of the reserve given by (4) we get that

$$\theta^*(t) = \frac{1}{1-\gamma} \frac{\alpha - r}{\sigma^2} (X^*(t) + g(t)),$$

which we recognize as (24).

Finally, plug in (22)–(24) into (4) to obtain

$$\begin{aligned} d(X^*(t) + g(t)) &= \left( r + \mu(t) + \frac{1}{1-\gamma} \left( \frac{\alpha - r}{\sigma} \right)^2 - \left( 1 + \mu(t) K_1^{\frac{1}{1-\gamma}} \right) \frac{1}{f(t)} \right) (X^*(t) + g(t)) dt \\ &\quad + \frac{1}{1-\gamma} \frac{\alpha - r}{\sigma} (X^*(t) + g(t)) dW(t). \end{aligned} \quad (36)$$

We get the solution

$$\begin{aligned} X^*(t) + g(t) &= (x_0 + g(0)) \exp \left\{ \int_0^t \left( r + \mu(s) + \frac{\frac{1}{2} - \gamma}{(1-\gamma)^2} \left( \frac{\alpha - r}{\sigma} \right)^2 - \frac{1 + \mu(s) K_1^{\frac{1}{1-\gamma}}}{f(s)} \right) ds \right. \\ &\quad \left. + \frac{1}{1-\gamma} \frac{\alpha - r}{\sigma} W(t) \right\}. \end{aligned} \quad (37)$$

In particular, since  $f$  is bounded away from zero,  $\forall t \in [0, T]$ , we have that  $X^*(t)$ ,  $t \in [0, T]$ , is well-defined, and clearly (9) is fulfilled. Moreover, since  $X^* + g$  is lognormally distributed one can easily show that

$$E^Q \left[ \int_0^T (\theta^*)^2 dt \right] < \infty,$$

which ensures that (10) is fulfilled. We conclude that the candidate optimal strategy  $(c^*, D^*, \theta^*)$  given by (22)–(24) is admissible. Since  $x_0$  and  $g(0)$  cannot both be zero we get, from (22), (23) and (37), that  $c^*(t), D^*(t) > 0$ ,  $t \in [0, T]$ , and  $X^*(T) > 0$ , and we note that the additional condition (16) is fulfilled, i.e.  $(c^*, \theta^*, D^*)$  is feasible. We conclude that  $(c^*, \theta^*, D^*)$  is the optimal strategy for the unrestricted problem (15).  $\square$

### 3.3 Comments on the unrestricted solution

The function  $g$  defined by (25) is referred to as the *actuarial present value of future labor income*. It is actuarial practice to define the reserve as a conditional expected present value of future payments. Isolating  $X$  in (11) leaves us with such an expression for the reserve. We get,  $\forall t \in [0, T]$ ,

$$X(t) = E^Q \left[ \int_t^T e^{-\int_t^s (r+\mu(y))dy} (c(s) - \ell(s) + \mu(s)D(s))ds + e^{-\int_t^T (r+\mu(y))dy} X(T) \middle| \mathcal{F}^W(t) \right].$$

We call the sum of the reserve and the actuarial present value of future labor income, given by (37), the *total reserve*. By (24) we have that the optimal amount invested in the stock is, comparable to Merton (1971), proportional to the total reserve and to the Sharpe ratio of the market, and dependent on the policyholder's risk aversion. The optimal consumption rate and the optimal life insurance strategy, given by (22) and (23), equals the total reserve divided by the deterministic function  $f$  given by (26), and for the optimal life insurance, multiplied by the weight factor  $K_1^{1/(1-\gamma)}$ . The function  $f$  can be interpreted as a subjective value of a unit consumption rate until terminal time  $T$  or time of death, whatever occurs first, plus a subjective actuarial present value of an endowment insurance paying  $K_1^{1/(1-\gamma)}$  upon death before time  $T$  and  $K_2^{1/(1-\gamma)}$  upon survival until time  $T$ . The value of  $f$  is subjective since the time dependent function  $\tilde{r}$  given by (27), used for discounting in the expression of  $f$ , depends on the policyholder's risk aversion and time preferences. Naturally, we conclude that the optimal consumption strategy is decreasing in  $K_1$  and  $K_2$ . That is, a policyholder to whom bequest and/or terminal pension are more important will optimally consume less. We also observe, that a more impatient policyholder, i.e. a policyholder with a greater time preference parameter  $\beta$ , optimally consumes more and buys more life insurance than a less patient policyholder. The optimal consumption and life insurance strategy depends in a much more complex way on the risk aversion parameter  $\gamma$ . By use of (36) we get, as also obtained in Steffensen and Kraft (2008), that,  $\forall t \in [0, T]$ ,

$$\begin{aligned} dc^*(t) &= \frac{d(X^*(t) + g(t))}{f(t)} - \frac{X^*(t) + g(t)}{f(t)^2} f'(t)dt \\ &= \left( \frac{r - \beta(t) + \left(\frac{\alpha-r}{\sigma}\right)^2}{1-\gamma} + \frac{1}{2} \frac{\gamma}{(1-\gamma)^2} \left(\frac{\alpha-r}{\sigma}\right)^2 \right) c^*(t)dt + \frac{1}{1-\gamma} \frac{\alpha-r}{\sigma} c^*(t)dW(t). \end{aligned}$$

We get,  $\forall t \in [0, T]$ , the solutions

$$\begin{aligned} c^*(t) &= c^*(0) \exp \left\{ \int_0^t \left( \frac{r - \beta(s) + \frac{1}{2} \left(\frac{\alpha-r}{\sigma}\right)^2}{1-\gamma} \right) ds + \frac{1}{1-\gamma} \frac{\alpha-r}{\sigma} W(t) \right\}, \\ D^*(t) &= D^*(0) \exp \left\{ \int_0^t \left( \frac{r - \beta(s) + \frac{1}{2} \left(\frac{\alpha-r}{\sigma}\right)^2}{1-\gamma} \right) ds + \frac{1}{1-\gamma} \frac{\alpha-r}{\sigma} W(t) \right\}. \end{aligned}$$

Note that  $c^*(t)$  and  $D^*(t)$  do not depend on  $\mu(t)$ . The optimal consumption and life insurance strategy depend only on  $\mu$  through  $f(0)$  and  $g(0)$  (through  $c^*(0)$  and  $D^*(0)$ ), i.e. on the overall level of  $\mu$ . Further discussions of the qualitative behavior of the optimal controls are found in Pliska and Ye (2007).

## 4 The restricted pension problem

From (37) we have for the unrestricted optimal solution that

$$X^*(t) > -g(t), \quad \forall t \in [0, T],$$

where  $g(t)$  is the time- $t$  actuarial value of future labor income given by (25). Allowing for a negative reserve is in conflict with the usual constraint in life insurance that the reserve should be non-negative at any time. This should be the case since, in practice, at any time the policyholder holds the right to stop paying pension contributions. We say that the policyholder holds the right to surrender. This serves to separate, institutionally, the pension business from the lending business. For further discussion of this issue see the Introduction, Section 1. As pointed out by Nielsen and Steffensen (2008), solving the unrestricted pension problem (15) we assume that the policyholder sticks to the pension contribution plan stipulated in the policy. The interpretation is that in the unrestricted pension problem given by (15) the policyholder decides initially on a consumption strategy, together with a life insurance strategy and an investment strategy, and is thereafter obligated to follow that specific strategy, i.e. he does not hold the right to surrender at any time.

The purpose of this paper is to find the optimal consumption, investment and life insurance strategy when the reserve is restricted to fulfill a capital guarantee at any point in time. As discussed, a capital guarantee of size zero is simply motivated by the policyholder's right to surrender at any time during the saving period. Often pension companies offer a minimum constant rate guarantee to the policyholders. In fact, such a guarantee is often found to be mandatory in compulsory (employer) pension schemes. In that case, the pension company guarantees a minimum rate of return on all pension contributions. Of course, no pension company can guarantee a constant minimum rate of return larger than the risk free short rate plus the objective mortality rate<sup>4</sup> since such a guarantee cannot be fulfilled with certainty. To cover a hole class of capital guarantee problems, including the two cases discussed, we allow the capital guarantee restriction to depend on the past in a very general way. Note that since future pension contributions may be stochastic, the future capital guarantee may become stochastic. For a comprehensive discussion and motivation of the problem see the Introduction, Section 1. More specific, consider the problem given by the indirect utility function

$$\sup_{(c, \theta, D) \in \mathcal{A}'} E \left[ \int_0^T e^{-\int_0^t (\beta(s) + \mu(s)) ds} \left( u(c(t)) + K_1 \mu(t) u(D(t)) \right) dt + K_2 e^{-\int_0^T (\beta(s) + \mu(s)) ds} u(X(T)) \right], \quad (38)$$

under the capital guarantee restriction

$$X(t) \geq k(t, Z(t)), \quad \forall t \in [0, T], \quad (39)$$

where  $Z(t) := \int_0^t h(s, X(s)) ds$ , and  $k$  and  $h$  are deterministic functions. The two guarantees discussed above are covered by

$$k(t, z) = 0, \quad (40)$$

and

$$k(t, z) = x_0 e^{\int_0^t (r^{(g)} + \mu(s)) ds} + e^{\int_0^t (r^{(g)} + \mu(s)) ds} z, \quad (41)$$

with  $h(s, x) = e^{-\int_0^s (r^{(g)} + \mu(y)) dy} (\ell(s) - c(s, x) - \mu(s)D(s, x))$ , where  $r^{(g)} \leq r$  is the minimum rate of return guarantee excess of the objective mortality rate  $\mu$ . Note that we in the expression

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<sup>4</sup>Remember that the mortality rate is given by a deterministic function, i.e. the pension company has no risk concerning life times.

of  $h$  have indicated the possible  $x$  dependence in the consumption and life insurance strategy. We can write (41) as

$$k(t, Z(t)) = x_0 e^{\int_0^t (r^{(g)} + \mu(y)) dy} + \int_0^t e^{\int_s^t (r^{(g)} + \mu(y)) dy} [\ell(s) - c(s) - \mu(s)D(s)] ds. \quad (42)$$

To clarify things, one could preferably compare (42) with the retrospective reserve expression given by (5).

The solution to the restricted capital guarantee problem given by (38) and (39) turns out to be of the Option Based Portfolio Insurance (OBPI) form. OBPI strategies are also obtained in El-Karoui et al. (2005) and Kronborg (2011), and for a European capital constraint in Teplá (2001). The big differences, compared to El-Karoui et al. (2005) and Kronborg (2011), are the introduction of life insurance and the possible *stochastic* capital guarantee restriction. In addition, El-Karoui et al. (2005) does not include consumption and labor income. Combining classic results from the American option theory in a Black-Scholes market and the OBPI construction technique we are able to solve the restricted optimization problem given by (38) and (39).

Before we present the results we need some notation: Still, denote by  $X^*$ ,  $c^*$ ,  $\theta^*$  and  $D^*$  the optimal unrestricted reserve and the unrestricted optimal consumption, investment and life insurance strategy derived in Subsection 3.2, respectively. Let  $Y^*(t) := X^*(t) + g(t)$  denote the optimal unrestricted total reserve. By (36) and (7) we have that

$$dY^*(t) = \left( r + \mu(t) - \left( 1 + \mu(t)K_1^{\frac{1}{1-\gamma}} \right) \frac{1}{f(t)} \right) Y^*(t)dt + \frac{1}{1-\gamma} \frac{\alpha - r}{\sigma} Y^*(t)dW^Q(t), \quad t \in (0, T), \quad (43)$$

$$Y^*(0) = y_0,$$

where  $y_0 := x_0 + g(0)$ . Denote by  $P_{y,z}^a(t, T, k + g)$  the time- $t$  value of an American put option with strike price  $k(s, Z(s)) + g(s)$ ,  $\forall s \in [t, T]$ , where  $Z(t) = z$ , and maturity  $T$  written on a portfolio  $Y$ , where  $Y$  is the solution to (43) which equals  $y$  at time  $t$ . By definition the price of such a put option is given,  $\forall t \in [0, T]$ , by

$$P_{y,z}^a(t, T, k + g) := \sup_{\tau \in \mathcal{T}_{t,T}} E^Q \left[ e^{-\int_t^\tau (r + \mu(s)) ds} (k(\tau, Z(\tau)) + g(\tau) - Y(\tau))^+ \mid Y(t) = y, Z(t) = z \right],$$

where  $\mathcal{T}_{t,T}$  is the set of stopping times taking values in the interval  $[t, T]$ .

**Theorem 4.1.** Consider the strategy  $(\widehat{c}, \bar{\theta}, \widehat{D})$  given by,  $\forall t \in [0, T]$ ,

$$\widehat{c}(t) := \lambda(t, Z(t))c^*(t) = \frac{\lambda(t, Z(t))Y^*(t)}{f(t)}, \quad (44)$$

$$\widehat{D}(t) := \lambda(t, Z(t))D^*(t) = \frac{\lambda(t, Z(t))Y^*(t)}{f(t)} K_1^{\frac{1}{1-\gamma}}, \quad (45)$$

$$\bar{\theta}(t) := \lambda(t, Z(t))\theta^*(t) = \frac{1}{1-\gamma} \frac{\alpha - r}{\sigma^2} \lambda(t, Z(t))Y^*(t), \quad (46)$$

combined with a position in an American put option written on the portfolio  $(\lambda(t)Y^*(t))_{s \in [t, T]}$  with strike price  $k(s, Z(s)) + g(s)$ ,  $\forall s \in [t, T]$ , and maturity  $T$ . Here  $\lambda$  is the increasing function defined by

$$\lambda(t, Z(t)) = \lambda(0, z_0) \vee \sup_{s \leq t} \left( \frac{b(s, Z(s))}{Y^*(s)} \right), \quad (47)$$

with  $\lambda(0, z_0)$  determined by

$$\lambda(0, z_0)(x_0 + g(0)) + P_{\lambda(0, z_0)y_0, z_0}^a(0, T, k + g) - g(0) = x_0, \quad (48)$$

and  $b$  being the optimal exercise boundary for the American put option given by

$$b(t, z) := \sup \{y : P_{y,z}^a(t, T, k + g) = (k(t, z) + g(t) - y)^+\}. \quad (49)$$

We have that

- The strategy is optimal for the restricted capital guarantee problem given by (38) and (39).

**Remark 4.1.** By (37) the optimal unrestricted total reserve,  $Y^*$ , is proportional to the initial total reserve  $y_0$ . Recall also that the optimal unrestricted consumption, life insurance and investment strategy, given by (22)–(24), are proportional to the total reserve. It should then be clear that starting out with reduced initial total reserve equal to  $\lambda(0, z_0)(x_0 + g(0))$  and following the strategy  $(\widehat{c}, \widehat{\theta}, \widehat{D})$  given by (44)–(49) results in a total reserve equal to  $\lambda(t, Z(t))Y^*(t)$ ,  $\forall t \in [0, T]$ . In total we get that the optimal restricted reserve becomes

$$\widehat{X}^{(\lambda)}(t) := \lambda(t, Z(t))Y^*(t) + P_{\lambda Y^*, Z}^a(t, T, k + g) - g(t), \quad t \in [0, T], \quad (50)$$

i.e. the optimal restricted reserve consists of a portfolio, an insurance on that portfolio and a loan equal to the actuarial value of future labor income.

**Remark 4.2.** The specific American put option is likely not to be sold in the market, but since the market is complete and frictionless such options can be replicated dynamically by a Delta-hedge, i.e. the optimal investment strategy can be written as

$$\widehat{\theta}(t) = \left(1 + \frac{\partial}{\partial y} P_{\lambda Y^*, Z}^a(t, T, k + g)\right) \bar{\theta}. \quad (51)$$

Since the put option has an opposite exposure to changes in the underlying portfolio  $(\lambda(t)Y^*(t))_{t \in [0, T]}$  we note that the total amount of money to invest in the risky asset,  $S$ , becomes smaller when we introduce an American capital guarantee to the control problem (15). One should notice, that at the boundary no risk is taken, i.e. the position in the American put option offsets the position in the underlying portfolio such that the total exposure to changes in the risky asset,  $S$ , becomes zero. Clearly, at the boundary it is optimal to consume a certain time-dependent part of the labor income, thereby leaving room for a risky position in  $S$  immediately after hitting the capital boundary.

*Proof of Theorem 4.1.* First, one should check that the strategy defined by Theorem 4.1 fulfills the capital guarantee restriction given by (39). Using that the value of an American put option is always greater than or equal to its inner value this is easily done. We get

$$\begin{aligned} \widehat{X}^{(\lambda)}(t) &:= \lambda(t, Z(t))Y^*(t) + P_{\lambda Y^*, Z}^a(t, T, k + g) - g(t) \\ &\geq \lambda(t, Z(t))Y^*(t) + [k(t, Z(t)) + g(t) - \lambda(t, Z(t))Y^*(t)]^+ - g(t) \\ &\geq k(t, Z(t)), \end{aligned} \quad (52)$$

Second, one should check that the strategy defined by Theorem 4.1 is admissible. It turns out that the rather complex strategy  $(\widehat{c}, \widehat{D}, \widehat{X}^{(\lambda)})$  involving the stochastic increasing function  $\lambda$  is the natural way to make the OBPI admissible. The heuristic argument is as follows. Consider the strategy  $(\widehat{c}, \widehat{D}, X^{(\lambda)})$  where  $\lambda(t, Z(t)) = \lambda_0, \forall t \in [0, T]$ , i.e. where we do not adjust  $\lambda$ . By (52) such a strategy clearly fulfills the capital guarantee restriction. The strategy corresponds to, in addition to follow the consumption, life insurance and investment strategy given by (44)–(46), to hold the initially bought American put option to maturity. In some scenarios this is to throw away money since the American put option is not sold when it is optimal to do so, i.e. the strategy is not admissible (self-financing). The complicated strategy defined in Theorem 4.1 is exactly designed to avoid this. The function  $\lambda$  defined by (47) and (48) corresponds to selling the American put option and re-balancing the strategy whenever optimal to do so. For a comprehensive explanation of the nature of the OBPI strategies see Kronborg (2011). A rigorous proof showing that the strategy in Theorem 4.1 is admissible is given in Appendix A.

To prove the optimality of the strategy given in Theorem 4.1 we consider an arbitrarily chosen feasible strategy  $(c, \theta, D)$  with corresponding reserve process  $(X(t))_{t \in [0, T]}$  satisfying  $X(0) = x_0$  and  $X(t) \geq k(t, Z(t)), \forall t \in [0, T]$ . Since  $u$  is a concave function we get that

$$\begin{aligned}
& \int_0^T e^{-\int_0^t (\beta(s) + \mu(s)) ds} [u(c(t)) + K_1 \mu(t) u(D(t))] dt + K_2 e^{-\int_0^T (\beta(s) + \mu(s)) ds} u(X(T)) \\
& - \left( \int_0^T e^{-\int_0^t (\beta(s) + \mu(s)) ds} [u(\widehat{c}(t)) + K_1 \mu(t) u(\widehat{D}(t))] dt + K_2 e^{-\int_0^T (\beta(s) + \mu(s)) ds} u(\widehat{X}^{(\lambda)}(T)) \right) \\
& = \int_0^T e^{-\int_0^t (\beta(s) + \mu(s)) ds} [u(c(t)) - u(\widehat{c}(t)) + K_1 \mu(t) (u(D(t)) - u(\widehat{D}(t)))] dt \\
& \quad + K_2 e^{-\int_0^T (\beta(s) + \mu(s)) ds} (u(X(T)) - u(\widehat{X}^{(\lambda)}(T))) \\
& \leq \int_0^T e^{-\int_0^t (\beta(s) + \mu(s)) ds} [u'(\widehat{c}(t))(c(t) - \widehat{c}(t)) + K_1 \mu(t) u'(\widehat{D}(t)) (D(t) - \widehat{D}(t))] dt \\
& \quad + K_2 e^{-\int_0^T (\beta(s) + \mu(s)) ds} u'(X(T)) (X(T) - \widehat{X}^{(\lambda)}(T)) \\
& =: (*). \tag{53}
\end{aligned}$$

Since  $(c, \theta, D)$  was arbitrarily chosen we simply end the proof by showing that  $E[(*)] \leq 0$ . By the CRRA property  $u'(xy) = u'(x)u'(y)$  we have

$$u'(\widehat{c}(t))(c(t) - \widehat{c}(t)) = u'(\lambda(t, Z(t)))u'(c^*(t))(c(t) - \widehat{c}(t)), \tag{54}$$

$$u'(\widehat{D}(t))(D(t) - \widehat{D}(t)) = u'(\lambda(t, Z(t)))u'(D^*(t))(D(t) - \widehat{D}(t)). \tag{55}$$

Observe that since  $Y^*(T) = X^*(T)$  the value of the terminal capital pension becomes

$$\begin{aligned}
\widehat{X}^{(\lambda)}(T) & = \lambda(T, Z(T))X^*(T) + [k(T, Z(T)) - \lambda(T, Z(T))X^*(T)]^+ \\
& = \max[\lambda(T, Z(T))X^*(T), k(T, Z(T))]. \tag{56}
\end{aligned}$$

By use of (56) and using that  $u'$  is a decreasing function we get that

$$\begin{aligned}
& u'(\widehat{X}^{(\lambda)}(T)) (X(T) - \widehat{X}^{(\lambda)}(T)) \\
& = \min[u'(\lambda(T, Z(T)))u'(X^*(T)), u'(k(T, Z(T)))] (X(T) - \widehat{X}^{(\lambda)}(T)) \\
& = u'(\lambda(T, Z(T)))u'(X^*(T)) (X(T) - \widehat{X}^{(\lambda)}(T)) \\
& \quad - [u'(\lambda(T, Z(T)))u'(X^*(T)) - u'(k(T, Z(T)))]^+ (X(T) - k(T, Z(T))),
\end{aligned}$$

where the last equality is established by using that  $\widehat{X}^{(\lambda)}(T) = k(T, Z(T))$  on the set  $\{(T, \omega) : u'(\lambda(T, Z(T))X^*(T)) \geq u'(k(T, Z(T)))\}$ . Since by assumption  $X(t) \geq k(t, Z(t)), \forall t \in [0, T]$ , we conclude that

$$u'(\widehat{X}^{(\lambda)}(T)) (X(T) - \widehat{X}^{(\lambda)}(T)) \leq u'(\lambda(T, Z(T)))u'(X^*(T)) (X(T) - \widehat{X}^{(\lambda)}(T)). \tag{57}$$



Now insert (54), (55) and (57) and then (30)–(32) into (53) to get

$$\begin{aligned}
E[(*)] &\leq E \left[ \int_0^T e^{-\int_0^t (\beta(s) + \mu(s)) ds} \left[ u'(\lambda(t, Z(t))) u'(c^*(t)) (c(t) - \widehat{c}(t)) \right. \right. \\
&\quad \left. \left. + K_1 \mu(t) u'(\lambda(t, Z(t))) u'(D^*(t)) (D(t) - \widehat{D}(t)) \right] dt \right. \\
&\quad \left. + K_2 e^{-\int_0^T (\beta(s) + \mu(s)) ds} u'(\lambda(T, Z(T))) u'(X^*(T)) (X(T) - \widehat{X}^{(\lambda)}(T)) \right] \\
&= \xi^* E^Q \left[ \int_0^T e^{-\int_0^t (r + \mu(s)) ds} u'(\lambda(t, Z(t))) \left( (c(t) - \widehat{c}(t)) + \mu(t) (D(t) - \widehat{D}(t)) \right) dt \right. \\
&\quad \left. + e^{-\int_0^T (r + \mu(s)) ds} u'(\lambda(T, Z(T))) (X(T) - \widehat{X}^{(\lambda)}(T)) \right].
\end{aligned}$$

Since  $u'(\lambda(t, Z(t)))$  is a decreasing function<sup>5</sup> we can use the integration by parts formula to get

$$\begin{aligned}
E[(*)] &\leq \xi^* \left( E^Q \left[ \underbrace{\int_0^T e^{-\int_0^t (r + \mu(s)) ds} u'(\lambda(t, Z(t))) \left( (c(t) - \widehat{c}(t)) + \mu(t) (D(t) - \widehat{D}(t)) \right) dt}_{(*)} \right. \right. \\
&\quad \left. \left. + \underbrace{\int_0^T u'(\lambda(t, Z(t))) d \left( e^{-\int_0^t (r + \mu(s)) ds} (X(t) - \widehat{X}^{(\lambda)}(t)) \right)}_{(**)} \right] \right. \\
&\quad \left. + E^Q \left[ \underbrace{\int_0^T e^{-\int_0^t (r + \mu(s)) ds} (X(t) - \widehat{X}^{(\lambda)}(t)) du'(\lambda(t, Z(t)))}_{(***)} \right] \right). \tag{58}
\end{aligned}$$

The third term in (58) can be rewritten as

$$\begin{aligned}
E^Q[(***)] &= E^Q \left[ \int_0^T e^{-\int_0^t (r + \mu(s)) ds} (X(t) - k(t, Z(t))) du'(\lambda(t, Z(t))) \right] \\
&\quad + E^Q \left[ \int_0^T e^{-\int_0^t (r + \mu(s)) ds} (k(t, Z(t)) - \widehat{X}^{(\lambda)}(t)) du'(\lambda(t, Z(t))) \right].
\end{aligned}$$

The first term is non-positive since per definition  $X(t) \geq k(t, Z(t))$ ,  $\forall t \in [0, T]$ , and  $du'(\lambda(t, Z(t))) \leq 0$ ,  $\forall t \in [0, T]$  ( $u'$  is decreasing and  $\lambda$  is increasing). The second term equals zero since  $du'(\lambda(t, Z(t))) \neq 0$  only on the set  $\{(t, \omega) : \widehat{X}^{(\lambda)}(t) = k(t, Z(t))\}$ . We conclude that  $E^Q[(***)] \leq 0$ . The two first terms of (58) can be rewritten as

$$E^Q[(*) + (**)] = E^Q \left[ \int_0^T u'(\lambda(t, Z(t))) dM_1(t) \right] - E^Q \left[ \int_0^T u'(\lambda(t, Z(t))) dM_2(t) \right], \tag{59}$$

where

$$\begin{aligned}
M_1(t) &:= \int_0^t e^{-\int_0^s (r + \mu(y)) dy} (c(s) + \mu(s) D(s) - \ell(s)) ds + e^{-\int_0^t (r + \mu(y)) dy} X(t), \\
M_2(t) &:= \int_0^t e^{-\int_0^s (r + \mu(y)) dy} (\widehat{c}(s) + \mu(s) \widehat{D}(s) - \ell(s)) ds + e^{-\int_0^t (r + \mu(y)) dy} \widehat{X}^{(\lambda)}(t).
\end{aligned}$$

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<sup>5</sup>This ensures the that the stochastic integral in (58) is well-defined.

Since both strategies are admissible we have by (13) that  $M_1$  and  $M_2$  are martingales under the equivalent measure  $Q$ . Since  $u'(\lambda(t, Z(t))) \leq u'(\lambda(0, z_0)), \forall t \in [0, T]$ , we get that

$$E^Q[(\star) + (\star\star)] = 0.$$

Finally, we can conclude that

$$E[(\star)] = E^Q[(\star) + (\star\star)] + E^Q[(\star\star\star)] \leq 0.$$

□

## 4.1 Numerical illustrations

In this subsection we illustrate the optimal strategy presented in Theorem 4.1 for the case of a minimum rate of return guarantee given by (42) with  $r^{(g)} = \frac{1}{2}r$ . The parameter values used for the simulations are, for the Black-Scholes market,  $r = 0.01885$ ,  $\alpha = 0.05885$  and  $\sigma = 0.2$ . The rather low risk free short rate and expected stock rate return should be interpreted as *inflation adjusted* parameters (subtracted 2.115 percent which is the average inflation in Denmark over the last 20 years). Adjusting for inflation allow us to directly compare the size of the terminal pension with past reserve values and labor income. The personal preference towards risk is set to  $\gamma = -4$  which is in accordance with the popular choice to have 20 percent of the reserve invested in stocks after retirement (we have  $\frac{1}{1-\gamma} \frac{\alpha-r}{\sigma^2} = 0.2$ ). The time preference is set to the natural value  $\beta = r$ , but one could of course equally well had chosen the time preference parameter to be smaller or greater than the risk free short rate. However, it seems natural to have  $\beta \geq r$  since this assures that one cannot, in a risk free way, obtain a greater amount of utility by simply investing in the risk free short rate and then consume at a later point in time. Finally, we have chosen to model the uncertain life time by a Gompertz-Makeham hazard rate,  $u(x) = \exp((x-m)/b)/b$ , with modal value and scale parameter as in Milevsky and Young (2007);  $(m, b) = (88.18, 10.5)$ .

The individual we consider has just turned 50 years old, has a reserve of size 200000 euro and a constant future labor income of  $\ell(t) = 30000$ ,  $t \in [50, T]$ , with terminal time  $T = 65$ . The weight factor  $K_2$  in (38) is calculated by use of (21), but since it seems natural that the reserve will also be invested after retirement we have chosen to use  $\bar{r} = 0.2\alpha + 0.8r = 0.02685$  (since  $\frac{1}{1-\gamma} \frac{\alpha-r}{\sigma^2} = 0.2$ ) for discounting in the expression of  $\bar{a}(T)$ . The weight factor  $K_1$  in (38) is set to  $K_1 = \int_0^{10} e^{-\int_t^{t+x} \beta(y)dy} dx \left(\bar{b}^{(10)}\right)^{-\gamma}$  with  $\bar{b}^{(10)} = \int_0^{10} e^{-\int_0^s r dy} ds$ . One can realize that this is in accordance with the standard Danish choice of a 10 years period certain pension to the inheritor starting at the time of death (if death occurs before terminal time  $T$ ).

The remaining 15 years of optimization has been discretized equidistantly into quarterly time points, i.e. into 60 time points. By this we mean that the guarantee is only valid quarterly and the American put option has been evaluated as a Bermuda put option with quarterly exercise dates. Likewise,  $\lambda$  and therefore the optimal consumption, investment and life insurance strategy has only been changed quarterly. Saying that, it is important to stress that between the quarterly time points, consumption, investments and bequest have been re-balanced continuously<sup>6</sup>.

It seems natural to compare the restricted solution with the unrestricted counterpart. Figure 1 illustrates how the restricted optimal reserve always is greater or equal to the guarantee while the unrestricted reserve sometimes is greater and sometimes smaller than the 'fictive' guarantee<sup>7</sup>. In fact we conclude that ignoring a guarantee requirement, and behaving optimal from the no guarantee point of view, will very likely result in a reserve smaller than the guarantee. Note that the restricted strategy can, for certain outcomes, outperform the unrestricted strategy both in terms of accumulated utility and terminal pension. In fact we have a 24 percent

<sup>6</sup>We have used (37).

<sup>7</sup>To be clear, there is of course no guarantee belonging to the unrestricted reserve, but we can calculate the guarantee anyway and then compare the unrestricted reserve with the guarantee.

probability that the restricted investor will obtain more utility over the interval of optimization than the unrestricted investor.

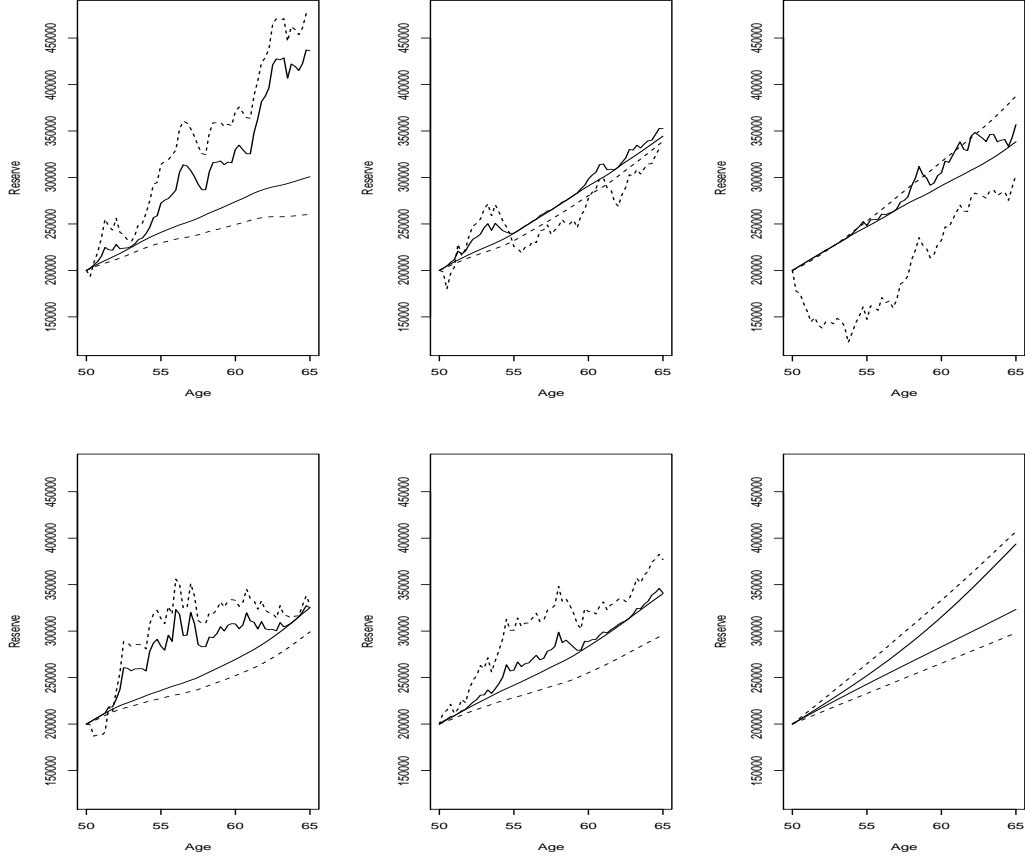


Figure 1: The restricted reserve (solid curves) and the unrestricted reserve (dashed curves) together with the corresponding guarantee (solid smooth curves) and fictive guarantee (dashed smooth curves), respectively. The lower right plot indicates the average scenario.

In Table 1 we see that the restricted optimal terminal pension has a slightly smaller median than the unrestricted one, but at the same time the restricted terminal pension has a much more narrow distribution. We also see that the pension saver with the minimum rate of return guarantee will, in contrast to the one without, more or less avoid ending up with a very small terminal pension. We want to stress that one cannot use Table 1 to compare the performances of two strategies since money spent before the terminal time  $T$  is not taking into account.

Fractal	0.001%	0.01%	2.5%	25%	median	75%	97.5%	99.99 %	99.999%
Restricted	10.21	10.58	11.33	12.18	13.12	14.30	17.08	22.03	23.60
Unrestricted	6.10	7.53	10.02	12.22	13.57	15.09	18.39	24.09	25.91

Table 1: Reserve at retirement expressed in numbers of yearly labor income (the terminal reserve divided by the labor income rate). Chosen fractals for both the restricted and unrestricted case are presented.

The actuarial value of future labor income is  $g(50) = 380387$  euro leaving the 50 years old investor with a total reserve of size  $Y(50) = 580387$ . By simulation we calculate  $\lambda(50) = 0.9041$  meaning that initially about 90 percent of the total reserve is invested in the optimal unrestricted portfolio and the remaining 10 percent is used to buy the American put option as an insurance against downfalls in the portfolio. Figure 2 shows how lambda increases at the time points where the insurance is in use (where it is optimal to sell the put option and re-calibrate the OBPI strategy accordingly to the new lambda value).

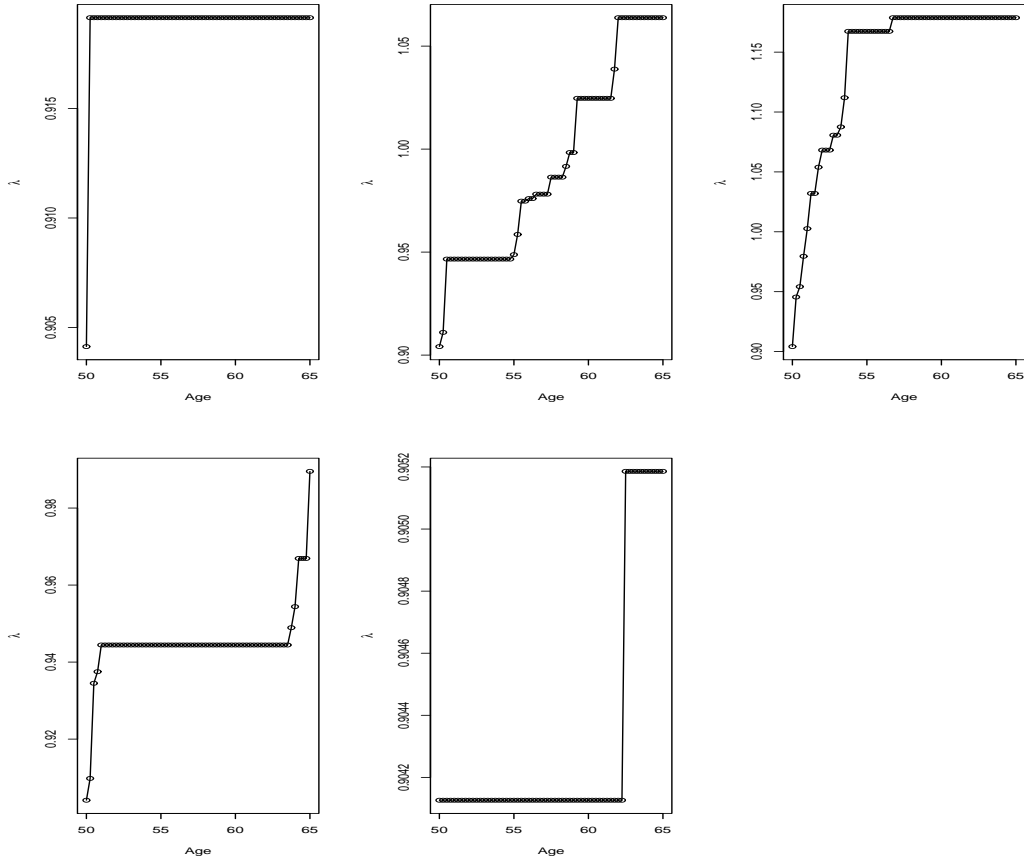


Figure 2: The lambda process corresponding to the reserve processes from Figure 1.

The optimal investment strategy is given by (51) and (46). Figure 3 shows the optimal fraction of the total reserve invested in stocks. The main observation to do is to note how wide the 95 percent confidence interval is. Even in the case of very big investment returns it takes about seven years (about half the optimization interval) before the insurance part becomes negligible and the restricted investor starts to invest (almost) as aggressively as the unrestricted investor. In the case of very low investment returns we see that the restricted investor gets stocked with the guarantee and cannot afford to take very much risk.

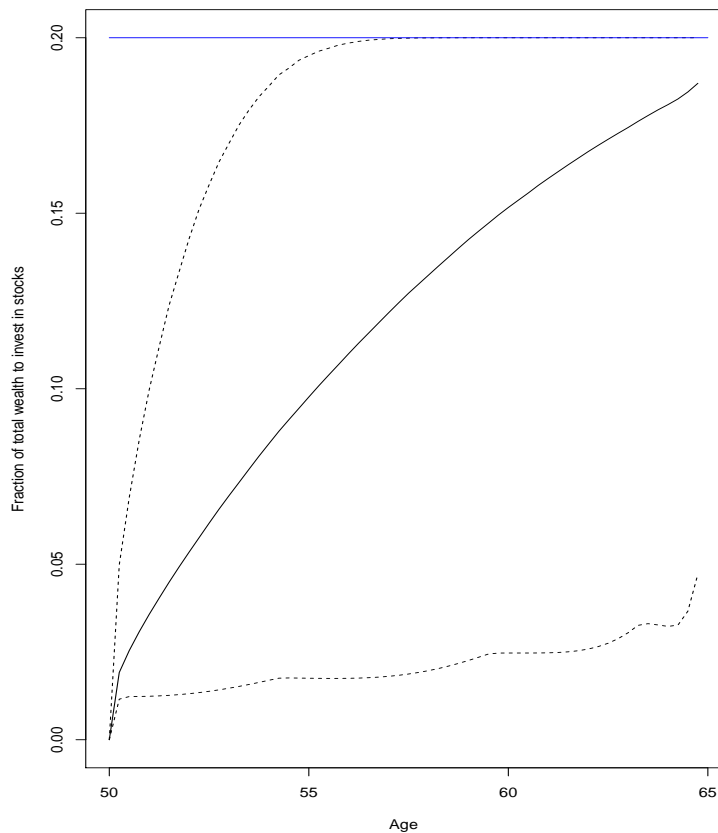


Figure 3: The restricted investment strategy presented as the fraction of the total reserve invested in stocks. The average scenario (solid curve) as well as the 95 percent confidence interval (dotted curves) are indicated together with the unrestricted case ( $\pi = 0.2$ ).

At last we have in Figure 4 illustrated the optimal life insurance and pension contributions as a fraction of the reserve and labor income, respectively. We see that at the age 50-55 the investor stops protecting parts of his future labor income and buys annuities instead. One should note how narrow the 95 percent confidence interval is in the restricted case compared to the unrestricted case. We also see that the restricted investor saves more money than the unrestricted investor. In other words: The introduction of a minimum rate of return guarantee seems to imply bigger pensions contributions. This is somehow surprising from a logical point of view but this is simply a consequence of the OBPI strategy.

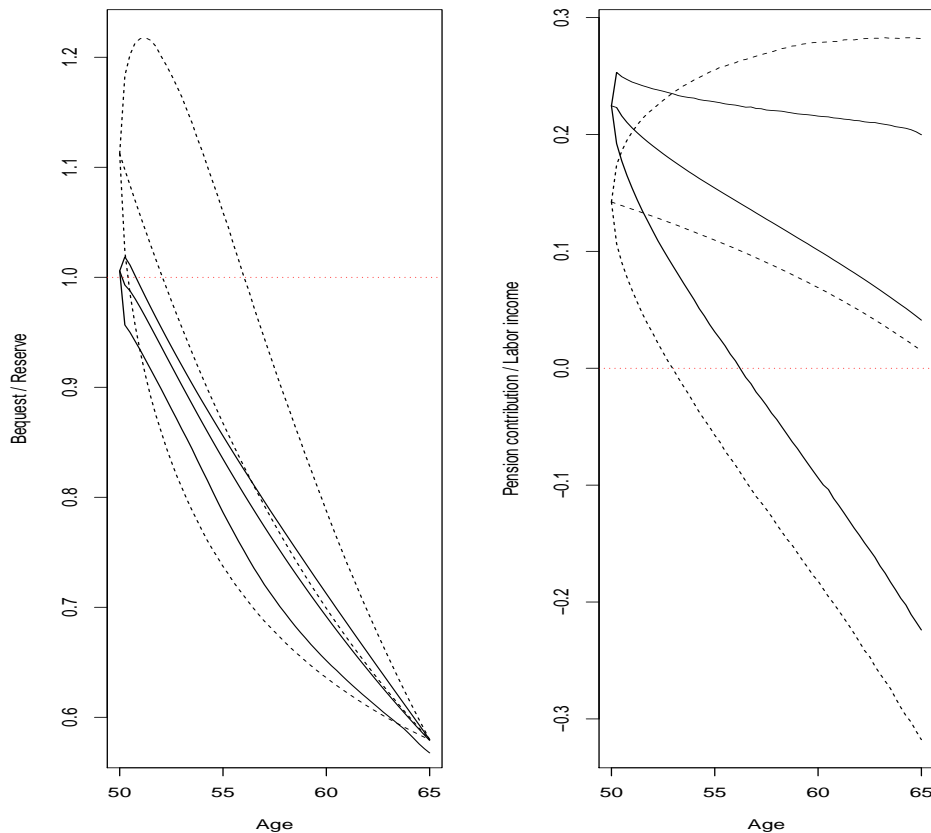


Figure 4: Left window: The restricted (solid curves) and unrestricted (dashed curves) life insurance strategy presented, for the average scenario, as the fraction of the reserve to be payed out upon death. The 95 percent confidence intervals are indicated. Right window: The restricted (solid curve) and unrestricted (dashed curve) pension contribution strategy presented, for the average scenario, as a fraction of the labor income rate. The 95 percent confidence intervals are indicated.

## 5 Conclusion

This paper solves the classic consumption, investment and life insurance control problem in the present of an American capital guarantee. Pension saving products often come with a minimum rate of return guarantee, which seems to be quiet a popular feature. Therefore the main focus, and the entire numerical section, has been on the solution for this specific type of capital guarantee. By use of cleaver chosen weight factors concerning the utility from consumption vs bequest and consumption vs pension the problem seems to be rather close to the real life question of how to save for retirement. The solution is an option based portfolio insurance (OBPI) strategy which is well know from the portfolio selection literature. However, to the authors knowledge this paper applies for the first time the OBPI approach to the saving for retirement problem. The encouraging realization offered by this paper is that the OBPI approach can handle the inclusion of a life insurance market as well as the introduction of a non-deterministic American guarantee, thereby allowing the capital guarantee to depend upon the size of the pension contributions.

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## A Appendix

*Proof of admissibility.* To prove that the strategy in Theorem 4.1 is admissible we first recall some basic properties of an American put option in a Black-Scholes market (see e.g. Karatzas and Shreve (1998)). We have

$$\begin{aligned} P_{y,z}^a(t, T, k + g) &= k(t, z) + g(t) - y, & \forall (t, y, z) \in \mathcal{C}^c, \\ \mathcal{L}P_{y,z}^a(t, T, k + g) &= (r + \mu(t))P_{y,z}^a(t, T, k + g), & \forall (t, y, z) \in \mathcal{C}, \\ \frac{\partial}{\partial y}P_{y,z}^a(t, T, k + g) &= -1, & \forall (t, y, z) \in \mathcal{C}^c, \end{aligned}$$

where (see (43))

$$\mathcal{L} := \frac{\partial}{\partial t} + \left( r + \mu - \left( 1 + \mu(t)K_1^{\frac{1}{1-\gamma}} \right) \frac{1}{f(t)} \right) y \frac{\partial}{\partial y} + h \frac{\partial}{\partial z} + \frac{1}{2} \left( \frac{1}{1-\gamma} \frac{\alpha - r}{\sigma} y \right)^2 \frac{\partial^2}{\partial y^2},$$

and

$$\mathcal{C} := \{(t, y, z) : P_{y,z}^a(t, T, k + g) > (k(t, z) + g(t) - y)^+\},$$

defines the continuation region. By  $\mathcal{C}^c$  we mean the complementary of  $\mathcal{C}$ , i.e. the stopping region. The continuation region can be described via the exercise boundary  $b$  given by (49). We get

$$\mathcal{C} = \{(t, y, z) : y > b(t, z)\}.$$

Introducing the function  $A$  by

$$A(t, y, z) := y + P_{y,z}^a(t, T, k + g) - g(t),$$

we can write (50) as

$$X^{(\lambda)}(t) = A(t, \lambda(t, Z(t))Y^*(t), Z(t)).$$



From the properties of  $P_{y,z}^a(t, T, k + g)$  we deduce that

$$\begin{aligned}
A(t, y, z) &= k(t, z), & \forall (t, y, z) \in \mathcal{C}^c, \\
\mathcal{L}A(t, y, z) &= \left( r + \mu - \left( 1 + \mu(t)K_1^{\frac{1}{1-\gamma}} \right) \frac{1}{f(t)} \right) y + (r + \mu(t))P_{y,z}^a(t, T, k + g) \\
&\quad - (-\ell(t) + (r + \mu(t))g(t)) \\
&= (r + \mu(t))A(t, y, z) + \ell(t) - \left( 1 + \mu(t)K_1^{\frac{1}{1-\gamma}} \right) \frac{y}{f(t)}, & \forall (t, y, z) \in \mathcal{C},
\end{aligned} \tag{60}$$

$$\mathcal{L}A(t, y, z) = \frac{\partial}{\partial t} k(t, z) + h(t, z) \frac{\partial}{\partial z} k(t, z), \quad \forall (t, y, z) \in \mathcal{C}^c, \tag{61}$$

$$\frac{\partial}{\partial y} A(t, y, z) = 0. \quad \forall (t, y, z) \in \mathcal{C}^c. \tag{62}$$

Observe that since  $Y^*(t)$  is linear in its initial value,  $\forall t \in [0, T]$ , we have for a constant  $\lambda$  that  $\lambda Y^*(t)$  has the same dynamics as  $Y^*(t)$ ,  $\forall t \in [0, T]$ . We now get by use of Itô's formula, (60)–(61), (22)–(24), and the fact that  $\lambda$  increases only at the boundary, that<sup>8</sup>

$$\begin{aligned}
&dA(t, \lambda(t, Z(t))Y^*(t), Z(t)) \\
&= [dA(t, \lambda Y^*(t), Z(t))]_{|\lambda=\lambda(t, Z(t))} + Y^*(t) \frac{\partial}{\partial y} A(t, \lambda(t, Z(t))Y^*(t), Z(t)) d\lambda(t, Z(t)) \\
&= \lambda(t, Z(t)) \frac{\partial}{\partial y} A(t, \lambda(t, Z(t))Y^*(t), Z(t)) \sigma \theta^*(t) dW^Q(t) \\
&\quad + \left[ (r + \mu(t))A(t, \lambda(t, Z(t))Y^*(t), Z(t)) \right. \\
&\quad \left. + \ell(t) - \lambda(t, Z(t))c^*(t) - \mu(t)\lambda(t, Z(t))D(t)^* \right] 1_{(\lambda(t, Z(t))Y^*(t) > b(t, Z(t)))} dt \\
&\quad + \left[ \frac{\partial}{\partial t} k(t, Z(t)) + h(t, Z(t)) \frac{\partial}{\partial z} k(t, Z(t)) \right] 1_{(\lambda(t, Z(t))Y^*(t) \leq b(t, Z(t)))} dt \\
&\quad + Y^*(t) \frac{\partial}{\partial y} A(t, \lambda(t, Z(t))Y^*(t), Z(t)) 1_{(\lambda(t, Z(t))Y^*(t) = b(t, Z(t)))} d\lambda(t, Z(t)).
\end{aligned}$$

Since by (62)  $\frac{\partial}{\partial y} A(t, \lambda(t)Y^*(t), Z(t)) = 0$  on the set  $\{(t, \omega) : \lambda(t, Z(t))Y^*(t) = b(t, Z(t))\}$  this reduces to

$$\begin{aligned}
&dA(t, \lambda(t, Z(t))Y^*(t), Z(t)) \\
&= \lambda(t, Z(t)) \frac{\partial}{\partial y} A(t, \lambda(t, Z(t))Y^*(t), Z(t)) \sigma \theta^*(t) dW^Q(t) \\
&\quad + [(r + \mu(t))A(t, \lambda(t, Z(t))Y^*(t), Z(t)) + \ell(t) - \lambda(t, Z(t))c^*(t) - \mu(t)\lambda(t, Z(t))D(t)^*] dt \\
&\quad + \left[ \frac{\partial}{\partial t} k(t, Z(t)) + h(t, Z(t)) \frac{\partial}{\partial z} k(t, Z(t)) \right. \\
&\quad \left. - [(r + \mu(t))k(t, Z(t)) + \ell(t) - \lambda(t, Z(t))c^*(t) - \mu(t)\lambda(t, Z(t))D(t)^*] \right] 1_{(\lambda(t, Z(t))Y^*(t) \leq b(t, Z(t)))} dt.
\end{aligned}$$

<sup>8</sup>  $\frac{\partial}{\partial y}$  now means differentiating w.r.t. the second variable.

Finally, since  $\{(t, \omega) : \lambda(t, Z(t))Y^*(t) \leq b(t, Z(t))\} = \left\{(t, \omega) : \lambda(t, Z(t)) = \frac{b(t, Z(t))}{Y^*(t)}\right\}$  has a zero  $dt \otimes dP$ -measure we conclude that

$$\begin{aligned} & dA(t, \lambda(t, Z(t))Y^*(t), Z(t)) \\ &= [(r + \mu(t))A(t, \lambda(t, Z(t))Y^*(t), Z(t)) + \ell(t) - \lambda(t, Z(t))c^*(t) - \mu(t)\lambda(t, Z(t))D(t)^*] dt \\ & \quad + \lambda(t, Z(t)) \frac{\partial}{\partial y} A(t, \lambda(t, Z(t))Y^*(t), Z(t)) \sigma \theta^*(t) dW^Q(t), \end{aligned}$$

i.e. by (14) the strategy is admissible. □