

Long guarantees with short duration: The rolling annuity

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Abstract

We present a new type of with-profits annuities which offer lifelong, yet hedgeable, guarantees. The rolling annuity gives a minimum lifelong guarantee at the time of contribution complemented with a series of guaranteed increases prior to retirement. Importantly, the initial guarantee and the subsequent increases are all set at prevailing market rates and hence are not known in advance. The structure of the guarantee implies that, prior to the last increase, the liability is equivalent to a zero-coupon bond maturing at the next increase and can therefore easily be hedged in the financial markets. Furthermore, the short duration implies that the financial and regulatory value will (essentially) coincide. We show financial fairness and we derive the reserve and thereby the hedging strategy. We also consider longevity risk, the duration profile, and report on a simulation study of the real value of the final payout.

Keywords: With-profits annuity, reserving, hedging, Solvency 2, longevity risk, duration, real value.

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1 Introduction

Traditionally, life annuities come in two flavors. In a defined benefit (DB) scheme the retiree receives an inflation indexed benefit depending on number of years of service and wage while working. In a with-profits pension scheme the retiree receives a guaranteed minimum pension with the possibility of receiving additional benefits if the funding status of the fund allows. Both systems are designed to provide financial security in retirement by guaranteeing the beneficiary lifelong payments at or above a certain level.

However, a combination of faltering financial markets, increased life expectancy and, in some cases, generous promises have put the traditional systems under increasing pressure. It is now apparent that assumptions, once believed to be conservative, were in fact overly optimistic and that many of the original schemes are not sustainable. Moreover, the long-lasting fall in interest rates has increased the market value of guaranteed liabilities dramatically and put further pressure on life and pension insurance companies operating under market-value accounting standards, e.g. in the Netherlands, cf. OECD (2013).

In response to these challenges there has been a massive shift from DB schemes to defined contribution (DC) schemes with no guarantees, see e.g. Chapter 1 of Ang (2014). Also, the terms of with-profits contracts are being rewritten and unconditional guarantees are made conditional. In both cases the result is the same: financial and insurance risks once borne collectively, or by a sponsor, are being transferred to the individual member. The new products are in essence individual savings vehicles with no risk sharing among members. Typically the new schemes offer greater transparency and flexibility than the old schemes. However, these appealing features notwithstanding, the new products do not fulfill the fundamental purpose of the products they replace: a guaranteed retirement income.

In this paper we present a new type of with-profits annuity: the rolling annuity. Similar to the traditional with-profits annuity, the rolling annuity is based on lifelong guarantees and (collective) risk sharing of biometric risks. However, unlike the traditional with-profits annuity, the financial guarantees of the rolling annuity do not rely on conservative assumptions on future market returns. Rather, the financial guarantees of the rolling annuity can be replicated in the financial markets, i.e. the guarantees are hedgeable. The rolling annuity thus combines the main virtue of traditional annuities (lifelong guarantees) with the main virtue of current products (hedgeability).

With-profits (or participating) annuities are common in collective pension plans in e.g. Germany and the Nordic countries, cf. Maurer et al. (2013); Rocha et al. (2011). In the traditional with-profits annuity the guaranteed benefits are calculated using conservative assumptions on both mortality and investment returns (the technical interest rate). Premiums are invested in the capital markets and over time assets will accumulate in the plan. If mortality assumptions and the technical rate were indeed conservative the accumulated returns from investments will produce a surplus. Over time this surplus is returned to the members in the form of indexation, i.e. by the discretionary decision of the pension plan management to increase member benefits.

The role of the technical rate is to provide stable benefit payments by smoothing investment returns over time between members of the plan. A few years of bad investment returns do not translate into immediate benefit cuts but are absorbed by the fund and shared between generations. When markets recover returns will again stabi-

lize, assets will appreciate, and indexation will resume. This all works well under the assumption that investment returns in the long run exceeds the technical rate.

However, two decades of falling interest rates has challenged established views on long-term returns and raised concerns about the sustainability of many pension plans. The problems are aggravated by the fact that under the terms of the traditional with-profits contract the technical rate is effective during the entire contract period. Thus, even though technical rates have decreased over the years, many old contracts with technical rates substantially above current market rates are still in effect.

The proposed rolling annuity also guarantees lifelong benefits, but on different terms. In the accumulation phase returns are guaranteed for a limited period only, e.g. 15 years, after which returns are guaranteed for another limited period and so on. The last period covers the entire payout phase. The guarantees are lifelong in the sense that guaranteed returns cannot be revoked, i.e. the guaranteed pension can only increase over time. However, prior to retirement the guaranteed return for the next period is not known in advance. To ensure hedgeability return guarantees are adapted to reflect market rates at the time of issuance. In essence, the rolling annuity replaces the technical rate with a (fair) market rate over successive periods.

The annual benefit guaranteed by the rolling annuity is based on a best estimate life expectancy, i.e. a life expectancy with no safety margin. Thus, in contrast to the traditional with-profits contract, mortality assumptions and return guarantees are both set at (a priori) realistic, rather than conservative, levels. The lifelong guarantee entails market and longevity risk which must be covered by risk capital provided by the pension plan. In this paper we develop the rolling annuity in the context of a collective pension fund, and we assume that the risk capital is provided by the members themselves by deductions from their contributions. Over time the deducted contributions are returned to the members in the form of indexation; this aspect is similar to the profit sharing mechanism of the with-profits contract.

The forthcoming Solvency 2 regulatory framework in the European Union requires tight risk management of life and pension insurance companies to ensure that issued guarantees are honored. The framework is devised by the European Insurance and Occupational Pensions Authority (EIOPA) with the aim of protecting policyholders and supporting the stability of the financial system.

A central part of the regulation concerns the construction of the discount curve used to value pension liabilities. At the heart of the construction lies a “risk free” interest rates term structure which extrapolates market data by assuming convergence to a so-called ultimate forward rate (UFR) determined by EIOPA. The UFR is set to 4.2% to reflect long-term assumptions about growth and inflation.¹

Convergence to an UFR and other intricacies of the Solvency 2 curve imply that it is ill-suited for hedging purposes. Specifically, the discounted value of a long-dated liability does not correspond to the value of a financial hedge, i.e. the true risk free price. In fairness, this is only partly a problem of Solvency 2. Liability cash flows extending beyond the horizon of the longest trading assets cannot be fully hedged in the markets, and hence cannot be assigned a unique risk free price. Thus, even using extrapolation schemes better suited for valuation and approximate hedging purposes,

¹EIOPA has recently announced a review of the methodology used to derive the UFR. Since the UFR is currently considerably higher than long-dated market rates, a review is likely to result in a downward adjustment.

some non-marketed residual risk will remain. In this perspective, the valuation discussion is merely a symptom of the non-hedgeability of traditional pension guarantees.

The guarantees of the rolling annuity are hedgeable by construction. As we show later on, in the accumulation phase the guarantees can be replicated by a sequence of zero-coupon bonds expiring at the end of the period for which the different return guarantees are in effect. Approaching retirement, the return guarantees are extended to the payout phase and at this stage a small fraction of the total guarantees will in fact be long-dated. However, for all practical purposes hedging requires trading in liquid, short duration bonds only, hence the title of the paper. Consequently, the Solvency 2 value and the market value will effectively coincide for a rolling annuity.

Rolling annuities have been implemented at the nationwide Danish pension scheme ATP. From 1 January 2015 members acquire (rolling annuity) guarantees for 80 pct. of their contributions, while the rest enters collectively owned free funds. The free funds provide the necessary capital to cover longevity risk of the guarantees and market risks of a large return seeking portfolio. Over time, the free funds are returned to the members in the form of indexation.

Rolling annuity guarantees are intended to form part of a with-profits contract, and the implementation at ATP represents one possibly product. Many other profit-sharing mechanisms and other sources for providing risk capital (if relevant) are of course possible. However, to convey the general idea we focus in this paper on the guarantees themselves, rather than on specific implementation choices.

Academic interest in life insurance contracts has primarily focused on pricing and hedging the embedded options under various capital market models. In particular, variable annuities with guaranteed minimum withdrawal benefit (GMWB) riders, participating policies with minimum interest rate guarantees and guaranteed annuity options have attracted attention, see e.g. Bacinello et al. (2014); Ng and Li (2013); Hyndman and Wenger (2014); Zaglauer and Bauer (2008); Grosen and Jørgensen (2000); Biffis and Millosovich (2006) and references therein.

Although we also consider valuation and hedging, the main contribution of the present paper is the annuity design itself. In this respect the paper is closer in spirit to the (smaller) literature on designing annuities with desirable properties. Properties previously considered in the literature include smooth benefit streams, inflation protection and mitigation of longevity risk, see e.g. Bruhn and Steffensen (2013); Guillén et al. (2006); Tiong (2013); Denuit et al. (2011); Richter and Weber (2011). To the knowledge of the authors, the specific goal of long guarantees hedgeable in shorter dated instruments is new.

We assume throughout that both prices and reserves are calculated on continuously updated best estimate mortality assumptions. No explicit price is charged to cover longevity risk, but at ATP this is in practice paid for by (part of) the 20 pct. deduction in contributions. This construction is only possible in a collective pension scheme which allows intergenerational risk sharing. For discussions on pros and cons of collective schemes, including intergenerational fairness issues, we refer the interested reader to Binsbergen et al. (2014); Gabay and Grasselli (2012); Kryger (2011, 2010). We finally remark that longevity risk could have been mitigated differently, e.g. by applying the group self-annuitization scheme of Piggot et al. (2005), see also Donnelly (2015). However, these adaptive schemes do not conform with our aim of offering guaranteed benefits, but they could well be of interest in other situations.

The rest of paper is organized as follows. After establishing notation in Section 2, the rolling annuity is developed in Section 3. This is the main section where we consider pricing, fairness, reserving and computational aspects. This is followed by Section 4 on longevity risk as quantified by the Solvency 2 stress, and Section 5 on the interest rate sensitivity (or duration) of the reserve. In Section 6 we report on a simulation study investigating the performance and risk profile of the rolling annuity in real terms. Finally, Section 7 concludes.

2 Preliminaries

We implicitly assume that the guarantees are issued by a pension fund or a life and pension insurance company which provides capital to cover risks associated with longevity and hedging. We further assume that the fund or company operates under fair value accounting standards.

As mentioned in the Introduction, the regulatory value under Solvency 2 of long-dated pension liabilities generally do not coincide with their financial value. However, the short duration of rolling annuities implies that the regulatory and financial value are typically close. For this reason, we calculate the reserve under the assumption that it equals the financial value of the guarantees.

The paper describes a mathematically idealized version of guarantees actually issued. For notational convenience we make the benign assumptions that benefit streams are continuous rather than in monthly installments, and that mortality is unisex rather than gender-specific. We also neglect a number of more substantial real world issues, e.g. costs, tax, multiple interest rate markets, and liquidity. These issues are of practical importance but outside the scope of the current presentation.

2.1 Notation

We assume the existence of a frictionless, default-free fixed income market, and we denote by $p_t(T)$ the time t price of a zero-coupon bond (also known as a discount bond) delivering for certain one monetary unit at maturity $T \geq t$. We further assume that the price can be expressed as

$$p_t(T) = E^Q \left[e^{-\int_t^T r_s ds} | \mathcal{F}_t \right], \quad (1)$$

where E^Q denotes expectation with respect to the so-called risk-neutral measure, see e.g. Bjrk (2009), r_s is the instantaneous risk-free rate at time s , and \mathcal{F}_t is the σ -field generated by the financial market up to time t .

To meet its obligations the fund must set aside a reserve based on prevailing market rates and a *best estimate* of the (future) force of mortality of its members. We denote by $\mu_{t,b}(s)$ the time t best estimate of the force of mortality at time s for a person born at time $b \leq s$. The notation reflects the fact that the estimate changes over time. In particular, the time t estimate of the probability that a person born at time b and alive at time u ($\geq b$) survives to time $T \geq u$ is given by

$$S_{t,b}(T|u) = e^{-\int_u^T \mu_{t,b}(s) ds}. \quad (2)$$

Similarly, the time t estimate of the expected number of years a person born at time b and alive at time u lives after time $T \geq u$ is given by

$$e_{t,b}(T|u) = \int_T^\infty S_{t,b}(w|u)dw = \int_T^\infty e^{-\int_u^w \mu_{t,b}(s)ds} dw. \quad (3)$$

Note that for all $u \leq T \leq P$ we have the relation

$$e_{t,b}(P|u) = S_{t,b}(T|u)e_{t,b}(P|T). \quad (4)$$

3 The rolling annuity

The purpose of the rolling annuity is to provide lifelong guaranteed benefits hedgeable in shorter dated securities. This is achieved by structuring the guarantee as a series of shorter (return) guarantees. To illustrate the idea consider a person making a contribution 35 years prior to retirement. The benefit level initially guaranteed depends on the expected number of years in retirement and the return which can be locked in for the next 15 years, say. After 15 years the guaranteed benefit level is increased by a factor corresponding to the return which can at that time be locked in for an additional 15 years. After 30 years the guaranteed benefit level is increased for the last time. This time the increase factors in the returns which can be locked in over the (expected) payout phase.

The initial guarantee is based on a best estimate of the expected number of years in retirement. The initial life expectancy assumption is guaranteed in the sense that the subsequent benefit increases do not depend on the future life expectancy evolution. Hence, the “risk” of life expectancy improvements in excess of those initially assumed is borne by the pension provider.

Rolling annuity guarantees are intended to exist alongside a return-seeking portfolio for two reasons. First, from a return perspective it is important to hold a well-diversified portfolio with broad exposure to market factors. The guarantees provide security but are exposed only to interest rate markets. Hence guarantees should be complemented with exposure to e.g. stocks, credit, commodities and inflation. Second, rolling annuity guarantees entail both longevity risk and hedging risk (although limited) and hence the guarantees can in general apply to only part of the contribution.

3.1 Tariff and guaranteed increases

For the purpose of this presentation, we consider a person born at time b with time of retirement P .² The (guarantee) contribution paid at time t ($< P$) is denoted by g_t . As the notation indicates, this is the contribution to which the guarantee pertains. The guaranteed minimum level of pension at time u associated with a contribution paid at time t is denoted by $z_t(u)$, for $u \geq t$. The pension consists of a continuous benefit stream of rate z from time of retirement to death. Since we measure time in years, z corresponds to the amount received per year. Finally, we denote by L the length of the period between successive guaranteed increases.

As an example illustrating the notation assume that $L = 15$ years, and consider a contribution paid at $t_0 = P - 35$, i.e. 35 years prior to retirement. The initial guarantee

²In practice, the age at which you are entitled to receive pension might change over time. We ignore this complication in the present exposition.

is based on the price of a zero-coupon bond with maturity 15 years, and the expected number of years in retirement. After 15 years the guarantee is increased by the return which can be obtained for certain over the coming 15 years. At the end of the second period, i.e. 5 years before retirement, the guarantee is increased by the return which can be obtained over the payout phase. If we let $t_1 = t_0 + 15$ and $t_2 = t_0 + 30$, the guaranteed minimum level of pension associated with the contribution paid at t_0 is given by

$$z_{t_0}(u) = \begin{cases} Z_0 & \text{for } t_0 \leq u < t_1, \\ Z_{15} & \text{for } t_1 \leq u < t_2, \\ Z_{30} & \text{for } t_2 \leq u, \end{cases}$$

where

$$Z_0 = \frac{g_{t_0}}{p_{t_0}(t_1)e_{t_0,b}(P|t_0)}, \quad Z_{15} = \frac{Z_0}{p_{t_1}(t_2)}, \quad Z_{30} = Z_{15} \frac{e_{t_2,b}(P|t_2)}{\int_P^\infty p_{t_2}(w)S_{t_2,b}(w|t_2)dw}.$$

Note that the increment factor used in Z_{30} can be interpreted as a weighted average of the returns which can be locked in over the payout phase, since $e_{t_2,b}(P|t_2) = \int_P^\infty S_{t_2,b}(w|t_2)dw$. From a theoretical point of view it could be argued that we should use the original mortality assumptions, given by $e_{t_0,b}$ and $S_{t_0,b}$. However, from a practical point of view it is preferable to use the *future* best estimates, given by $e_{t_2,b}$ and $S_{t_2,b}$, since this allows us to work with only one set of assumptions at any given instant in time.³

The increment factors used to obtain Z_{15} and Z_{30} are known only after 15 and 30 years, respectively. From a financial point of view and disregarding longevity risk, the guarantee corresponds to a so-called floater with a reset frequency of 15 years in the accumulation phase.

In order to state the general expression for the guarantee we need some additional notation. Denote by I_t the set consisting of t and the time points at which the guarantee associated with a contribution paid at time t is subsequently increased (if any), i.e. $I_t = \{t + iL : i = 0, 1, 2, \dots\} \cap [t, P)$. Further, let $I_t(u) = I_t \cap [t, u]$. Using this notation the minimum guarantee at time u associated with a contribution paid at time t is given by

$$z_t(u) = \frac{g_t}{e_{t,b}(P|t)} \prod_{\tau \in I_t(u)} \xi_\tau^{-1} \quad \text{for } u \geq t, \quad (5)$$

where

$$\xi_\tau = \begin{cases} p_\tau(\tau + L) & \text{for } \tau < P - L, \\ \int_P^\infty p_\tau(w)S_{\tau,b}(w|\tau)dw / e_{\tau,b}(P|\tau) & \text{for } P - L \leq \tau < P. \end{cases} \quad (6)$$

We can interpret (5) as follows. First, the (guarantee) contribution is divided according to the length of the payout phase, i.e. the expected number of years in retirement. This

³Note that the increment factor used in Z_{30} is generally increasing in life expectancy, since the longer dated returns are then assigned a larger weight relative to the shorter dated returns. This implies that if life expectancy evolves faster than expected, such that $e_{t_2,b}(P|t_2) > e_{t_0,b}(P|t_2)$, then the guaranteed pension will generally be *larger* than it would have been under the original assumptions.

is the affordable level in the absence of returns. Second, the guarantee is increased with the returns that can be obtained for certain over successive periods. In total the periods cover the entire period from the time of contribution to the last benefit.⁴

It follows from (5) that the *initial* minimum guarantee associated with a contribution paid at time $t < P$ is given by

$$z_t(t) = \frac{g_t}{p_t(t+L)e_{t,b}(P|t)} \quad \text{for } t < P - L, \quad (7)$$

and

$$z_t(t) = \frac{g_t}{\int_P^\infty p_t(w)S_{t,b}(w|t)dw} \quad \text{for } P - L \leq t < P. \quad (8)$$

In the first case there will be one or more guaranteed increases before retirement, while in the second case the guaranteed level is fixed.

3.2 Pension credits

Due to the structure of the guarantee it is at first glance necessary to keep track of the size and timing of all (guarantee) contributions. Assume that contributions are paid at the series of time points $t_0 < t_1 < \dots < t_N < P$. Then the total minimum guarantee at time u is given by

$$z(u) \equiv \sum_{t_j \leq u} z_{t_j}(u) = \sum_{t_j \leq u} \frac{g_{t_j}}{e_{t_j,b}(P|t_j)} \prod_{\tau \in I_{t_j}(u)} \xi_\tau^{-1}, \quad (9)$$

where ξ_τ is given by (6). Note that $z(u)$ is constant for $u \geq P$ since all guaranteed increases occur prior to retirement. Hence in the payout phase it is sufficient to keep track of the single number $z(P)$.

In practise, however, contributions are paid at regular intervals, e.g. monthly or annually, and this can be utilized to reduce the size of the record we have to keep in order to calculate the guarantee. Assume that $t_j = t_0 + j$ for $j = 1, \dots, N$, i.e. contributions are paid once a year (or, equivalently, all contributions paid in a given year are subject to the same tariff). Further, assume that L is an integer, e.g. $L = 15$. We can then exploit that guarantees associated with contributions paid L years apart will be subject to the *same* subsequent increases.

We introduce a *vector of pension credits* of length L , $(z^0(u), \dots, z^{L-1}(u))$. The components decompose the total guarantee by the time elapsed since the guarantee was last increased (or the contribution was made). The vector is initialized at t_0 by setting

$$z^0(t_0) = z_{t_0}(t_0), \quad z^i(t_0) = 0 \quad \text{for } i = 1, \dots, L - 1. \quad (10)$$

At each subsequent contribution time, t_j , the vector is updated as follows

$$z^0(t_j) = z^{L-1}(t_{j-1})\xi_{t_j}^{-1} + z_{t_j}(t_j), \quad z^i(t_j) = z^{i-1}(t_{j-1}) \quad \text{for } i = 1, \dots, L - 1. \quad (11)$$

⁴In the actual product it is also guaranteed that $z_t(u)$ can never decrease, i.e. $z_t(u) = \frac{g_t}{e_{t,b}(P|t)} \prod_{\tau \in I_t(u)} \max\{\xi_\tau^{-1}, 1\}$. The members are thus insured against negative rates. In this presentation we consider only the “pure” product with no option elements.

That is, the guarantee associated with the current contribution is recorded in the first component together with the guarantees due for increase. The rest of the vector is “shifted” one place to the right to reflect the fact that one year has passed. Note that the vector has to be updated every year until retirement, even if no contribution is made, i.e. $g_{t_j} = 0$. It is easy to verify that $z(u) = \sum_{i=0}^{L-1} z^i(u)$ for all $u \geq t_0$. Hence, the total guarantee can be calculated from a record of only L numbers.

3.3 Financial fairness

Loosely speaking the guarantee is financially fair in that the members receive a fair return for the period the money is in the fund. From a financial fairness point of view it makes no difference whether the guarantee is given in full initially, or as a series of guaranteed increases, as long as it reflects the current market conditions. In the following we make these statements precise.

Strictly speaking, the guarantee is only financially fair under certain idealized assumptions. First, the guaranteed annuity level is fixed at the guaranteed increase preceding retirement. At this point the pension fund guarantees an annuity cash flow which in principle extends well beyond maturities for which an interest rate market can be claimed to exist. For the purpose of showing fairness we assume nevertheless that the entire cash flow can be replicated in the market. From a practical perspective only a small fraction of the cash flow will be (very) long dated, so this is not cause for concern.

Second, the actual rolling annuity contains a “longevity guarantee”, in the sense that the guaranteed pension is essentially unaffected by the future life expectancy evolution. A part of the contribution is used to cover this, and other, risks. We return to this issue in Section 4, but here we disregard longevity risk and instead assume that the best estimate mortality and the realized mortality coincide.

Financial fairness amounts to showing that the discounted value of the expected benefit cash flow equals the (guarantee) contribution. Under the simplifying assumptions made above we need to show that for fixed t

$$g_t = E^Q \left[z_t(P) \int_P^\infty e^{-\int_t^w r_s ds} S_{t,b}(w|t) dw | \mathcal{F}_t \right]. \quad (12)$$

Using (5) of Section 3.1 this is equivalent to showing

$$1 = \frac{1}{e_b(P|t)} E^Q \left[\prod_{\tau \in I_t(P)} \xi_\tau^{-1} \int_P^\infty e^{-\int_t^w r_s ds} S_b(w|t) dw | \mathcal{F}_t \right], \quad (13)$$

where we have omitted the time subscript of e_b and S_b to indicate that, by assumption, the mortality estimate does not change over time.

Let us consider the case where $I_t(P) = \{t, t_1, t_2\}$ with $t_1 = t + L$ and $t_2 = t + 2L$. Recall, that by construction $t < t_1 < P - L \leq t_2 < P$. By (6) we have

$$\xi_t = p_t(t_1), \quad \xi_{t_1} = p_{t_1}(t_2), \quad \xi_{t_2} = \frac{S_b(t_2|t)}{e_b(P|t)} \int_P^\infty p_{t_2}(w) S_b(w|t_2) dw, \quad (14)$$

where the last equality uses (4) and that mortality assumptions are constant over time. Now, by using first (14) and the relation $S_b(w|t) = S_b(t_2|t) S_b(w|t_2)$, next the tower

property and linearity of conditional expectations, and finally repeated use of (1) we can rewrite the right-hand side of (13) as

$$\begin{aligned}
& E^Q \left[\frac{e^{-\int_t^{t_1} r_s ds} e^{-\int_{t_1}^{t_2} r_s ds} \int_P^\infty e^{-\int_{t_2}^w r_s ds} S_b(w|t_2) dw}{p_t(t_1) p_{t_1}(t_2) \int_P^\infty p_{t_2}(w) S_b(w|t_2) dw} \Big| \mathcal{F}_t \right] \\
&= E^Q \left[\frac{e^{-\int_t^{t_1} r_s ds}}{p_t(t_1)} E^Q \left[\frac{e^{-\int_{t_1}^{t_2} r_s ds} \int_P^\infty E^Q [e^{-\int_{t_2}^w r_s ds} | \mathcal{F}_{t_2}] S_b(w|t_2) dw}{\int_P^\infty p_{t_2}(w) S_b(w|t_2) dw} \Big| \mathcal{F}_{t_1} \right] \Big| \mathcal{F}_t \right] \\
&= E^Q \left[\frac{e^{-\int_t^{t_1} r_s ds}}{p_t(t_1)} E^Q \left[\frac{e^{-\int_{t_1}^{t_2} r_s ds}}{p_{t_1}(t_2)} \Big| \mathcal{F}_{t_1} \right] \Big| \mathcal{F}_t \right] = E^Q \left[\frac{e^{-\int_t^{t_1} r_s ds}}{p_t(t_1)} \Big| \mathcal{F}_t \right] = 1.
\end{aligned}$$

We conclude that (disregarding longevity) rolling annuity guarantees are financially fair. The general case can be proved similarly.

3.4 Reserving

We assume that the pension fund must at all time hold a (market value) reserve equal to the financial value of the guarantee. We also assume that the reserve is based on the *current* best estimate of future mortality. In the following we calculate the reserve for a person born at time b and still alive.

Recall that I_t is the set consisting of t and the times at which the guarantee associated with a contribution paid at time t is subsequently increased (if any). For any two numbers a and b we let $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. By definition the reserve at time u for the guarantee associated with a contribution paid at time t is given by

$$V_t(u) \equiv E^Q \left[\tilde{z}_{u,t}(P) \int_{P \vee u}^\infty e^{-\int_u^w r_s ds} S_{u,b}(w|u) dw \Big| \mathcal{F}_u \right] \quad \text{for } u \geq t, \quad (15)$$

where

$$\tilde{z}_{u,t}(P) = \frac{g_t}{e_{t,b}(P|t)} \prod_{\tau \in I_t} \tilde{\xi}_{u,\tau}^{-1}, \quad (16)$$

with

$$\tilde{\xi}_{u,\tau} = \begin{cases} p_\tau(\tau + L) & \text{for } \tau < P - L, \\ \int_P^\infty p_\tau(w) S_{u \wedge \tau, b}(w|\tau) dw / e_{u \wedge \tau, b}(P|\tau) & \text{for } P - L \leq \tau < P. \end{cases} \quad (17)$$

Note that $\tilde{\xi}_{u,\tau}$ equals ξ_τ of (6), except for the mortality estimates used. Prior to τ , the former uses the current (time u) mortality estimate while the latter uses the future (time τ) estimate. The point being that the mortality estimate used in the final guaranteed increase is not known prior to the increase. Hence, although $z_t(P)$ of (5) is the pension actually received, the reserve will be calculated on the basis of $\tilde{z}_{u,t}(P)$ which uses the current mortality estimate. After the final increase the two quantities are equal.

Proposition 3.1. Consider a fixed t . Assuming the person is alive, the reserve at $u \geq t$ associated with a contribution paid at time t is given by

$$V_t(u) = \begin{cases} z_t(u)e_{u,b}(P|u)p_u(\tau_N(u)) & \text{for } u < \tau_F, \\ z_t(u) \int_{P \vee u}^{\infty} p_u(w)S_{u,b}(w|u)dw & \text{for } u \geq \tau_F, \end{cases} \quad (18)$$

where $\tau_F = \max I_t$ denotes the time of the final increase and, for $u < \tau_F$, $\tau_N(u) = \min[I_t \cap (u, P)]$ denotes the time of the next increase.

Proof. Consider first the case $u \geq \tau_F$. After the final increase $z_t(\cdot)$ is constant and equal to $\tilde{z}_{u,t}(P)$. The result then follows immediately from (15) using (1).

Consider next the case $u < \tau_F$. Assume first that only the final increase remains, i.e. $u < \tau_N(u) = \tau_F$. We then have

$$\begin{aligned} V_t(u) &= E^Q \left[z_t(u) \tilde{\xi}_{u,\tau_F}^{-1} \int_P^{\infty} e^{-\int_u^w r_s ds} S_{u,b}(w|u) dw \middle| \mathcal{F}_u \right] \\ &= z_t(u) E^Q \left[E^Q \left[e_{u,b}(P|\tau_F) \frac{\int_P^{\infty} e^{-\int_u^w r_s ds} S_{u,b}(w|u) dw}{\int_P^{\infty} p_{\tau_F}(w) S_{u,b}(w|\tau_F) dw} \middle| \mathcal{F}_{\tau_F} \right] \middle| \mathcal{F}_u \right] \\ &= z_t(u) E^Q \left[e^{-\int_u^{\tau_F} r_s ds} E^Q \left[\frac{e_{u,b}(P|u) \int_P^{\infty} e^{-\int_{\tau_F}^w r_s ds} S_{u,b}(w|u) dw}{S_{u,b}(\tau_F|u) \int_P^{\infty} p_{\tau_F}(w) S_{u,b}(w|\tau_F) dw} \middle| \mathcal{F}_{\tau_F} \right] \middle| \mathcal{F}_u \right] \\ &= z_t(u) e_{u,b}(P|u) E^Q \left[\frac{e^{-\int_u^{\tau_F} r_s ds} \int_P^{\infty} E^Q [e^{-\int_{\tau_F}^w r_s ds} | \mathcal{F}_{\tau_F}] S_{u,b}(w|\tau_F) dw}{\int_P^{\infty} p_{\tau_F}(w) S_{u,b}(w|\tau_F) dw} \middle| \mathcal{F}_u \right] \\ &= z_t(u) e_{u,b}(P|u) p_u(\tau_F), \end{aligned}$$

as claimed. Assume next that $k > 1$ increases remain. Denote the corresponding time points by τ_i , i.e. $\tau_N(u) = \tau_0 < \tau_1 < \dots < \tau_{k-1} = \tau_F$. By iterated conditioning and the result just shown we then have

$$\begin{aligned} V_t(u) &= z_t(u) e_{u,b}(P|u) E^Q \left[e^{-\int_u^{\tau_0} r_s ds} E^Q \left[\prod_{i=0}^{k-2} \frac{e^{-\int_{\tau_i}^{\tau_{i+1}} r_s ds}}{p_{\tau_i}(\tau_{i+1})} \middle| \mathcal{F}_{\tau_{k-1}} \right] \middle| \mathcal{F}_u \right] \\ &= \dots = z_t(u) e_{u,b}(P|u) E^Q \left[e^{-\int_u^{\tau_0} r_s ds} \middle| \mathcal{F}_u \right] = z_t(u) e_{u,b}(P|u) p_u(\tau_0). \end{aligned}$$

□

We note that in (18) the reserve is expressed in terms of the current value of the guarantee and the time of the next increase, if any. This is important from a practical perspective as it implies that the reserve can be calculated from the vector of pension credits, cf. Section 3.2, and current market data and mortality estimates.

Assume that we are at a point of increase and that at least one more increase remains, i.e. $u + L = \tau_N(u) \leq \tau_F$. Assume also that $e_{u-,b}(P|u) = e_{u,b}(P|u)$, i.e. that the mortality estimate does not change at u . We then have

$$V_t(u) = z_t(u) e_{u,b}(P|u) p_u(\tau_N(u)) = z_t(u-) e_{u,b}(P|u) = V_t(u-), \quad (19)$$

showing that the reserve is unaffected by the guaranteed increase. The same can be shown to be true at the final increase. Mathematically, the reserve is unchanged because the guarantee is increased by (the inverse of) the same factor used to discount the new guarantee.

3.5 Hedging

The pension fund receives contributions from a large number of people. Say that K_t persons all born at time b pay a contribution of g_t at time t , and let K_u denote the number of people still alive at time $u \geq t$. The fraction of the cohort surviving over time is termed the *realized* survival (probability), \tilde{S}_b . By definition, $K_u = \tilde{S}_b(u|t)K_t$. Realized survival is a stochastic quantity which depends on both the underlying force of mortality (systematic variability) and the random nature of death (unsystematic variability). By the law of large numbers, the realized survival is close to the underlying “true” survival probability for a large cohort of i.i.d. individuals, i.e. the unsystematic variability is negligible.

Consider first the idealized case where mortality assumptions are constant over time and equals realized mortality. In particular, we then have $e_{u,b}(P|u) = e_{t,b}(P|u)$ and $\tilde{S}_b(u|t) = S_{t,b}(u|t)$. Assume that $u < \tau_F$ and let us consider the total reserve set aside for the cohort at time u ,

$$\bar{V}_t(u) \equiv K_u V_t(u) = K_t z_t(u) p_u(\tau_{N(u)}) \tilde{S}_b(u|t) e_{u,b}(P|u) = K_t z_t(u) p_u(\tau_{N(u)}) e_{t,b}(P|t),$$

where the first equality uses the definition of realized survival and Proposition 3.1, while the second equality uses the relations $\tilde{S}_b(u|t) e_{u,b}(P|u) = S_{t,b}(u|t) e_{t,b}(P|u) = e_{t,b}(P|t)$. Thus, before the final increase the reserve evolves like a zero-coupon bond with principal $K_t z_t(u) e_{t,b}(P|t)$ expiring at the time for the next guaranteed increase, $\tau_{N(u)}$.

In practise, mortality assumptions are updated periodically. Also, realized survival since the last update will typically be close, but not exactly equal, to assumed survival. Let m denote the time for the latest mortality update prior to u . We then have $e_{u,b}(P|u) = e_{m,b}(P|u)$ and $\tilde{S}_b(u|m) \approx S_{m,b}(u|m)$, and thereby

$$\bar{V}_t(u) = K_m z_t(u) p_u(\tau_{N(u)}) \tilde{S}_b(u|m) e_{u,b}(P|u) \approx K_m z_t(u) p_u(\tau_{N(u)}) e_{m,b}(P|m).$$

Thus, between increases and between updates of mortality assumptions the total reserve evolves essentially like a zero-coupon bond expiring at the time for the next guaranteed increase. Hence, from the point of view of the pension fund the liability can be semi-statically hedged, in the sense that the hedge needs to be adjusted only every L years, or when mortality assumptions change markedly. Further, if L is kept at a duration where market liquidity is high, say, up to 20 years, the hedging can be done very efficiently.

After the final increase the liability takes on the duration of a fully guaranteed annuity cash flow. However, since the final increase occurs near retirement the majority of the resulting liability cash flow will be at durations that can easily be hedged, and only a small fraction will be very long dated.

Finally, we mention that the reserve can also be expressed as the discounted value of an expected benefit stream,

$$V_t(u) = \int_{P \vee u}^{\infty} p_u(w) E^w[\tilde{z}_{u,t}(P) | \mathcal{F}_u] S_{u,b}(w|u) dw \quad \text{for } u \geq t, \quad (20)$$

where E^w denotes expectation with respect to the so-called w -forward measure, see e.g. Chapter 26 of Bjrk (2009). Note that each expectation uses a different measure. The representation highlights the fact that members are indeed guaranteed lifelong benefits, although financially the duration only extends to the next guaranteed increase. The representation also offers an alternative route to the proof of Proposition 3.1, but we do not pursue that here.

4 Longevity risk

Members are guaranteed lifelong pensions at a certain level. The initial guarantee is calculated on the basis of a best estimate mortality forecast which includes expected future increases in life expectancy. However, there is a risk that the actual average life span will exceed the estimate, and that the pension fund therefore will have to pay out benefits for longer than initially assumed. This risk is termed longevity risk.

Longevity risk is borne by the pension fund, in the sense that the guaranteed increases are unaffected by the life expectancy evolution. More precisely, in a collective pension fund longevity risk is borne by the members collectively rather than individually. In practice, the mortality forecast is reestimated periodically, e.g. annually, and the reserve is adjusted accordingly. If life expectancy increases more than expected this will result in a reduction of collective free funds, which in turn implies lower future indexation.

Under the forthcoming Solvency 2 regulatory framework, the solvency capital requirement for life and pension insurance companies in the EU must take longevity risk into account. In the so-called Standard Formula longevity risk is quantified by a stress scenario in which mortality rates used for the calculation of technical provisions are decreased (uniformly) by 20%. In the following we will quantify longevity risk by this stress. It should be mentioned, however, that under Solvency 2 companies are also allowed to use (partial) internal models to calculate longevity risk, see Jarner and Møller (2015) for an example of a specific model.

For illustrative purposes and to obtain explicit results we will base the presentation on a standard Gompertz-Makeham mortality law.⁵ We consider as usual a given cohort born at time b . To aid interpretation we assume that $\mu_{t,b} = \mu_b$ for all t . Hence differences over time are due only to the increased age of the cohort. Let, for $u \geq b$

$$\mu_b(u) = Ae^{Bx} + C, \quad (21)$$

where $x = u - b$ is the age at time u , and $A = 1.5 \cdot 10^{-5}$, $B = 0.1$ and $C = 2 \cdot 10^{-4}$. The parameters are obtained as rounded estimates based on Danish mortality data for males and females combined in 2011 for ages 20–100.⁶

The probability that a person from the cohort survives to time T conditioned on being alive at time u is given by

$$S_b(T|u) = \exp \left\{ -C(y - x) - \frac{A}{B}(e^{By} - e^{Bx}) \right\}, \quad (22)$$

⁵In ATP best estimate mortality forecasts are obtained from the SAINT model, cf. Jarner and Kryger (2011), and longevity risk is calculated by an internal model using the stochasticity of the SAINT model.

⁶Data are retrieved from the Human Mortality Database, www.mortality.org. At the time of writing, 2011 is the latest year available.

where $y = T - b$ and $x = u - b$ is the age at time T and u , respectively. Further, it follows from Theorem 2 of Missov and Lenart (2013) and (4) that the remaining life expectancy after time T conditioned on being alive at time u can be expressed as

$$e_b(T|u) = \frac{1}{B} e^{A/B} \left(\frac{A}{B}\right)^{C/B} \Gamma\left(-\frac{C}{B}, \frac{A}{B} e^{Bx}\right) S_b^{-1}(u|b), \quad (23)$$

where $x = T - b$ is the age at time T and where $\Gamma(a, w) = \int_w^\infty t^{a-1} e^{-t} dt$ denotes the upper incomplete gamma function.

Finally, we denote by $\tilde{\mu}_b$ the ‘‘stressed’’ force of mortality, $\tilde{\mu}_b(u) = 0.8\mu_b(s)$. Similarly, we denote by \tilde{S}_b and \tilde{e}_b , respectively, the survival probability and remaining life expectancy calculated on the basis of $\tilde{\mu}_b$. These quantities can be calculated from (22) and (23) upon replacing A and C with, respectively, $\tilde{A} = 1.2 \cdot 10^{-5}$ and $\tilde{C} = 1.6 \cdot 10^{-4}$.

To illustrate the impact of the stress, we show in Table 1 the conditional remaining life expectancy at given ages under both μ_b and $\tilde{\mu}_b$, together with the absolute and relative increase.

Age (x)	0	25	55	75	100
Life expectancy under μ_b	81.60	57.06	28.58	12.85	2.42
Life expectancy under $\tilde{\mu}_b$	83.94	59.31	30.60	14.35	2.90
Absolute increase	2.34	2.25	2.02	1.50	0.48
Relative increase	2.9%	4.0%	7.08%	11.6%	19.9%

Table 1: Conditional remaining life expectancy in years at given ages under μ_b and $\tilde{\mu}_b$, i.e. $e_b(T|T)$ and $\tilde{e}_b(T|T)$ for $T = b + x$.

The quantity of interest is the relative increase of the reserve. To exploit the results above and to gain some general insights we will make the simplifying assumption that the discount factors are of the form $p_t(T) = e^{-(T-t)r}$ for some fixed r . Under these assumptions we have

$$p_u(w)S_b(w|u) = \exp\left\{-(C+r)(y-x) - \frac{A}{B}(e^{By} - e^{Bx})\right\} = S_b^r(w|u), \quad (24)$$

where $y = w - b$ and $x = u - b$, and S_b^r is given by (22) upon replacing C with $C^r = 2 \cdot 10^{-4} + r$; the corresponding ‘‘life expectancy’’ is denoted e_b^r .

Let $\tilde{V}_t(u)$ denote the reserve at time u associated with a contribution paid at time t and based on $\tilde{\mu}_b$. Using (24), it now follows from Proposition 3.1 that

$$\frac{\tilde{V}_t(u)}{\tilde{V}_t(u)} = \begin{cases} \tilde{e}_b(P|u)/e_b(P|u) & \text{for } u < \tau_F, \\ \tilde{e}_b^r(P \vee u|u)/e_b^r(P \vee u|u) & \text{for } u \geq \tau_F, \end{cases} \quad (25)$$

where $\tau_F = \max I_t$ denotes the time of the final increase and \tilde{e}_b^r is given by (23) upon replacing A and C with, respectively, $\tilde{A} = 1.2 \cdot 10^{-5}$ and $\tilde{C}^r = 1.6 \cdot 10^{-4} + r$.

Table 2 illustrates the impact of the Solvency 2 stress. The table shows the relative reserve increase at different ages when the interest rate is 0%, 2% and 4%. The reserve corresponds to a single premium paid at a given age, and the stress is applied at the

time of payment. It is assumed that the age of retirement is $P = 65$ years, and that the period between successive increases is $L = 15$ years.

At younger ages when one or more increases remain, in this case below age 50, the reserve increase depends only on the expected number of years in retirement. At age 45, for instance, expected years in retirement increase from 18.4 years to 20.4 years, an increase of 11%. Since survival until retirement is high, the expected number of years in retirement is almost the same for all younger ages, and consequently the reserve increase will be around 11% for all ages below age 50.

At higher ages when the pension is fixed, and perhaps already in payment, the reserve increase depends on the “discounted” (remaining) number of years in retirement. As the interest rate increases the role of mortality is reduced and, consequently, the relative reserve increase is reduced. In retirement, the reserve increase is determined primarily by the relative increase in remaining life expectancy, cf. Table 1. The lowest relative reserve increase is found for members about to retire. This is due to a combination of high expected number of years in retirement, and reduced sensitivity due to discounting.

Rate (r)	Age (x)						
	25	45	55	65	75	85	100
0%	11.4%	11.0%	10.5%	9.0%	11.6%	14.9%	19.9%
2%	11.4%	11.0%	8.7%	7.3%	10.0%	13.5%	19.1%
4%	11.4%	11.0%	7.3%	5.9%	8.7%	12.3%	18.3%

Table 2: Relative reserve increase when applying the Solvency 2 stress of a 20% mortality reduction. The quantity shown is $\tilde{V}_T(T)/V_T(T) - 1$ for $T = b + x$, corresponding to a single premium paid at age x . Age of retirement is $P = 65$ years, and period between increases is $L = 15$ years.

5 Duration

In the absence of an external sponsor, a collective pension fund can only issue truly guaranteed pensions (to all its members) if the liabilities can be financially hedged. For theoretical considerations we often assume that (zero-coupon) bonds at all maturities are traded. In practice, however, liability hedging at a large scale is possible only for maturities up to around 30 years. After that point the market is shallow and illiquid. In fact, due to the limited issuance of long-dated government bonds, market depth and liquidity are decreasing already for maturities of 15 to 20 years.

A hedgeable lifelong guarantee is the main motivation for the product. In the following we consider the duration of the liability and demonstrate how it evolves over time. As an example we consider rolling annuity guarantees with a period between guaranteed increases of $L = 15$ years, and we show that only a small fraction of the total liability extends beyond 30 years at any time. Indeed, the majority of the liability cash flow is at most 15 years; hence concentrated at maturities where market liquidity is generally very high.

We let $r_t(T)$ denote the continuously compounded zero-coupon yield for the period $[t, T]$, defined by the relation $p_t(T) = \exp\{-(T - t)r_t(T)\}$. For fixed t , the yield as a

function of T is called the yield curve. The yield curve represents the returns that can be locked in at time t over different periods of time.

We define the *interest rate sensitivity* of a financial value by the (negative) relative change in value in response to a parallel shift of the yield curve. The interest rate sensitivity of (the price of) a zero-coupon bond is thus

$$-\frac{\partial p_t(T)}{\partial r} \frac{1}{p_t(T)} = T - t; \quad (26)$$

from which it follows that the interest rate sensitivity of the present value of a cash flow is equal to the average term to maturity of the payments (weighted by their present value). In the following we use the interest rate sensitivity of the reserve as a measure of the duration of the corresponding liability.⁷

By Proposition 3.1 and (26) the duration of the reserve at time $u \geq t$ associated with a contribution paid at time t is given by

$$D_t(u) \equiv -\frac{\partial V_t(u)/\partial r}{V_t(u)} = \begin{cases} \tau_N(u) - u & \text{for } u < \tau_F, \\ \frac{\int_{P \vee u}^{\infty} (w-u) p_u(w) S_{u,b}(w|u) dw}{\int_{P \vee u}^{\infty} p_u(w) S_{u,b}(w|u) dw} & \text{for } u \geq \tau_F, \end{cases} \quad (27)$$

where τ_F denotes the time of the final increase and, for $u < \tau_F$, $\tau_N(u)$ denotes the time of the next increase. We observe that between increases the duration equals the time that remains to the next increase, reflecting the fact that the reserve evolves like a zero-coupon bond. After annuitization at the final increase the duration is the average term of (expected) payments.

Over the course of their working life members pay contributions periodically, e.g. monthly or annually. The reserve thus consists of the sum of reserves corresponding to contributions paid at different times. To illustrate the effect of the aggregation of guarantees over time we consider the following simple setup.

Assume that a member pays contributions once a year from age 25 to age 64, i.e. 40 contributions in total. For ease of notation we assume that the person is born at $b = 0$. Assume that the first (guarantee) contribution is $g_{25} = 100$, and that contributions hereafter increase with 2% (continuously compounded) each year to reflect wage increases, i.e. $g_t = 100 \exp\{(t - 25)2\%\}$ for $t = 25, \dots, 64$. We assume further that the age of retirement is $P = 65$ years and that the period between guaranteed increases is $L = 15$ years. For illustrative purposes we assume that $r_t(T) = 3\%$ for all t and T . Finally, we assume that the mortality law is given by (21) of Section 4 and that $\mu_{t,b} = \mu_b$ for all t , i.e. we disregard longevity risk.

Table 3 illustrates how the guarantee build up over time. Guarantees issued for ages below 50 are subject to one or two guaranteed increases. For these ages the initial guarantee is based on the expected number of years in retirement and the price of a zero-coupon bond with 15 years to maturity. Since the expected number of years in retirement is almost constant the increase over time reflects primarily the increased contribution. From age 50 onwards a fixed-level annuity is guaranteed at the outset. The jump in “initial guarantee” from age 45 to age 50 is due to the longer duration of the liability and hence higher accumulated returns being factored in.

⁷Indeed, interest rate sensitivity is often referred to as duration. We follow this terminology and use the two terms interchangeably.

In Figure 1 the acquired guarantees are broken down into how much is guaranteed initially and how much is guaranteed at the subsequent increases. For contributions paid at ages 25–34 the initial guarantee constitutes less than half of the final guarantee, while for age 35–49 the initial guarantee constitutes between half and two thirds of the final guarantee. Of the final, accumulated guarantee about two thirds are due to initial guarantees.

Age (x)	25	30	35	40	45	50	55	60	65
Contribution	100	111	122	135	149	165	182	201	-
Years in retirement	18.1	18.1	18.2	18.3	18.4	18.6	18.8	19.3	20.1
Initial guarantee	8.7	9.6	10.5	11.6	12.7	19.5	18.2	16.9	-
Accumulated guarantee	8.7	55	105	166	254	362	523	707	831

Table 3: The table shows the contribution, the expected number of years in retirement at the time of contribution, $e_b(P|x)$, and the additional *initial* guarantee acquired at the time of contribution, i.e. $z_x(x)$ of Section 3.1. The last row shows the accumulated guarantee, including past guaranteed increases, i.e. $z(x)$ of Section 3.2. Note that by assumption $x = t$.

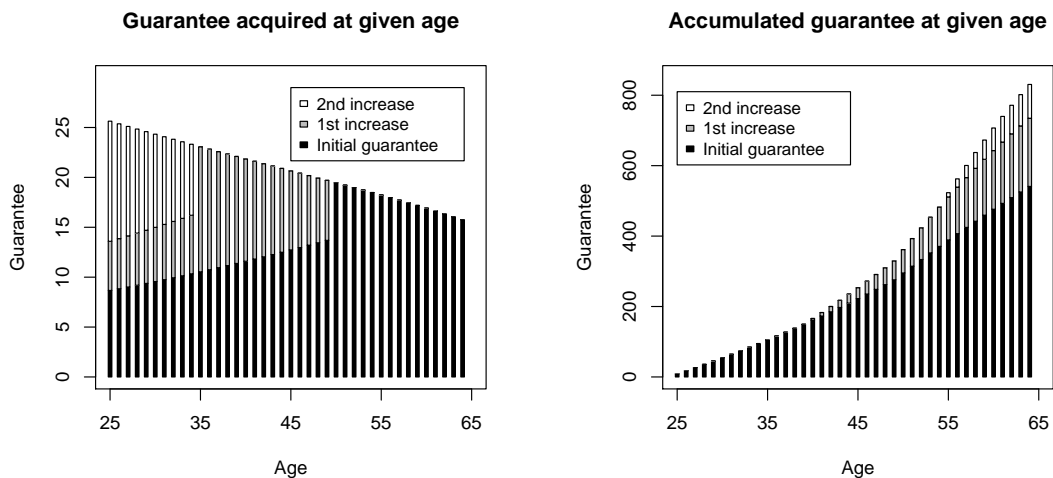


Figure 1: Left plot shows the guarantee acquired for the contribution paid at a given age and the right plot shows the accumulated guarantee at a given age. In both plots the guarantee is divided into the part initially guaranteed (black bar), the part guaranteed at the first increase (gray bar), and the part guaranteed at the second increase (white bar).

The total reserve and the duration of the total reserve are given by, respectively,

$$V(u) \equiv \sum_{t_j \leq u} V_{t_j}(u) \quad \text{and} \quad D(u) \equiv -\frac{\partial V(u)/\partial r}{V(u)} = \sum_{t_j \leq u} D_{t_j}(u) \frac{V_{t_j}(u)}{V(u)}, \quad (28)$$

where $t_j = 25 + j$ for $j = 0, \dots, 39$ are the contribution times. At each t_j the reserve jumps by the size of the contribution, $\Delta V(t_j) \equiv V(t_j) - V(t_j-) = g_{t_j}$; between contributions and in retirement the reserve evolves continuously according to the Thiele

differential equation

$$\frac{\partial V(u)}{\partial u} = \{r + \mu_b(u) - z(P)1_{[P;\infty)}(u)\}V(u) \text{ for } u \notin \{t_j : j = 0, \dots, 39\}, \quad (29)$$

where $z(P)$ denotes the final annuity level.

Figure 2 shows the evolution of V and D over time. While the reserve itself evolves as one would expect, the duration of the reserve evolves in a less intuitive manner. There are three distinct epochs. During the first 15 years the duration decreases from the initial value of 15 years to just over 7.5 years immediately before age 40. From age 40 to around age 60 the duration (generally) increases as the first guarantees are renewed and, from age 50, as fixed-level annuities are being issued. From age 60 onwards the duration decreases as retirement approaches and, after retirement, as a consequence of remaining life expectancy going to zero. We note that the guaranteed increases give rise to a “ripple effect” visible as kinks after 15 and 30 years. We also note that at no point in time is the duration of the reserve more than 15 years.

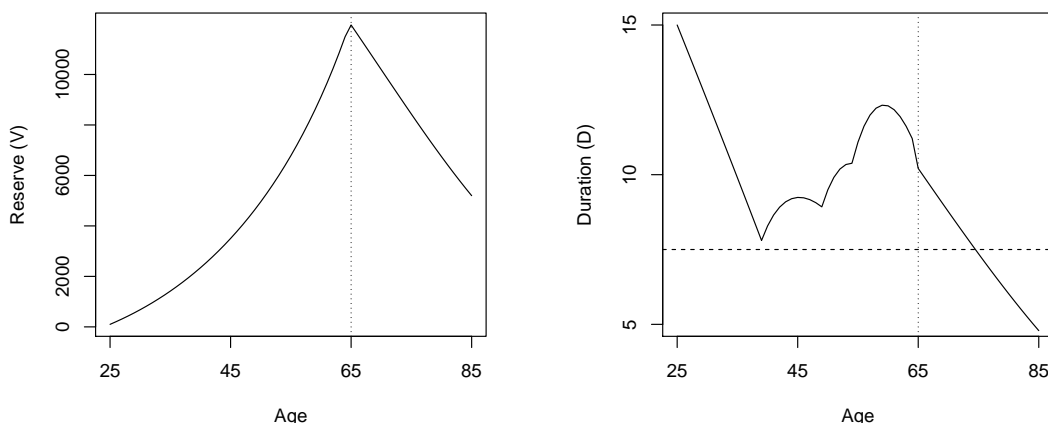


Figure 2: Left plot shows the total reserve and right plot shows the duration of the total reserve measured in years. The horizontal dashed line marks the “average time to next increase” of $L/2 = 7.5$ years. In both plots the vertical dotted line at age 65 marks the age of retirement.

The duration of the total reserve is an aggregate statistic which measures the average length of the cash flow. From a practical point of view it is also of interest to know the share of long-dated liabilities, since this is the part potentially problematic to hedge. The left plot of Figure 3 shows the reserve for liabilities with maturities longer than 30 years as a fraction of the total reserve.⁸ With a guarantee period of 15 years (the case considered so far) the “long-dated” reserve is non-negligible only in the period leading up to retirement. The “long-dated” reserve peaks at age 56 where it accounts

⁸More precisely, we plot the ratio $V^{30}(u)/V(u)$ as a function of u , where $V^{30}(u) \equiv \sum_{t_j \leq u} V_{t_j}^{30}(u)$ with $V_{t_j}^{30}(u) = z_{t_j}(u) \int_{u+30}^{\infty} p_u(w) S_b(w|u) dw$ for $u \geq \tau_F(t_j)$, and zero otherwise. The “long-dated” reserve, V^{30} , is the part of the reserve which uses discount factors with maturities of 30 years or longer. Note that the stated expression assumes that the guarantee period is less than 30 years. For illustrative purposes we use an interest rate of 3% for all maturities, i.e. $p_u(w) = \exp\{-(w-u)3\%\}$. In practise, however, it is not obvious how to value long-dated liabilities and determining proper long term interest rates is a topic of ongoing debate.

for just over 5% of the total reserve. At the time of retirement only 1.4% of the total reserve concerns payments with maturities longer than 30 years. The plot also shows that if the guarantee period is extended to 25 years the fraction of long-dated liabilities is more than twice as high. Note that since we are using constant, flat yield curves the total reserve is the same for all guarantee periods.

Finally, the right plot of Figure 3 shows the duration of the total reserve for guarantee periods of 5, 15 and 25 years. With a guarantee period of 25 years the duration is essentially decreasing throughout the contract, while for a guarantee period of 5 years the duration is approximately constant in most of the accumulation phase and rising towards the end. In contrast, with a guarantee period of 15 years the duration of the liability is of similar magnitude in the entire accumulation phase and in the early part of the payout phase. It might be argued that a flat duration profile is preferable since different cohorts then have similar hedging demands, i.e. their liabilities are equally “risky”.

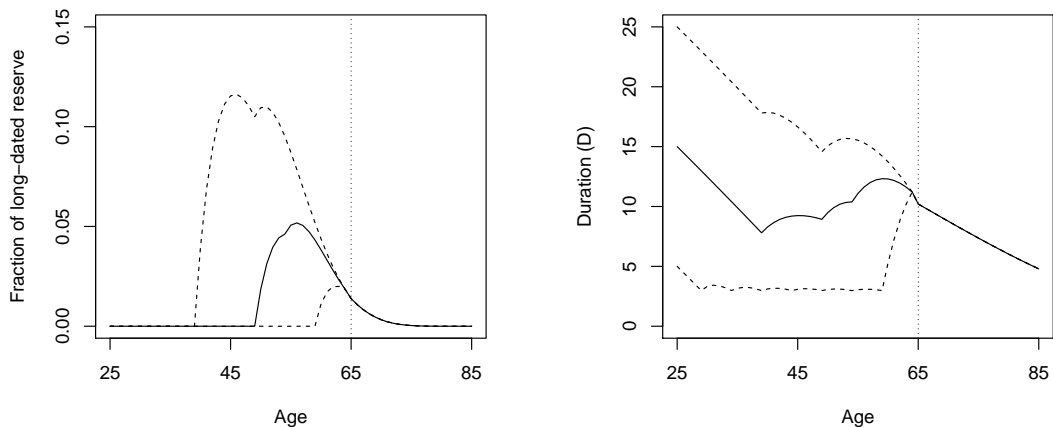


Figure 3: Left plot shows the reserve for maturities over 30 years as a fraction of the total reserve and right plot shows the duration of the total reserve measured in years. In both plots the solid line represents a guarantee period of 15 years, while the dashed lines below and above the solid lines represent guarantee periods of 5 and 25 years, respectively. The vertical dotted line at age 65 marks the age of retirement.

6 Simulation study

In the following we present a simulation study of how the performance of the rolling annuity depends on the length of the guarantee period. We measure the value of the final annuity in real terms since, arguably, this is the only yardstick of interest when it comes to pensions. The analysis is based on a joint stochastic model of interest rates and inflation dynamics also used by Munk et al. (2004).

Intuitively, with (nominal) yield curves being generally upwardly sloping, longer guarantees have the benefit of higher returns on average than shorter guarantees. However, shorter guarantees are less effected by a period of low interest rates and they sooner benefit from a subsequent rise in interest rates than longer guarantees “locking

in” low returns for a long time. In real terms, assuming (nominal) interest rates and inflation being correlated, we similarly expect longer guarantees to perform better on average than shorter guarantees. On the other hand, we expect shorter guarantees to track inflation more closely than longer guarantees. The length of the guarantee period thus represents a trade-off between higher average returns versus better adaptation to inflation.

6.1 Capital market model

We assume that the short (nominal) interest rate follows an Ornstein-Uhlenbeck process,

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma_r dW_t^r, \quad (30)$$

where \bar{r} is the long-run mean of the short interest rate, κ describes the degree of mean reversion, σ^r is the interest rate volatility, and W^r is a standard Brownian motion. Note that (30) describes the “real world” dynamics of the short interest rate — sometimes referred to as the P -dynamics as opposed to the Q -dynamics used for pricing.

We assume further that the term structure of interest rates (the yield curve) is of the form considered by Vasicek (1977). In particular, the price at time t of a zero-coupon bond maturing at time $T \geq t$ is given by

$$p_t(T) = \exp\{G(\Delta) - H(\Delta)r_t\}, \quad (31)$$

with $\Delta = T - t$,

$$H(\Delta) = \frac{1}{a} (1 - \exp\{-a\Delta\}), \quad (32)$$

$$G(\Delta) = \left(b - \frac{\sigma_r^2}{2a^2}\right) (H(\Delta) - \Delta) - \frac{\sigma_r^2}{4a} H^2(\Delta), \quad (33)$$

and where a and b are parameters controlling the slope and level of the yield curves.⁹

Recall that the continuously compounded zero-coupon yield for the period $[t, T]$, $r_t(T)$, is defined by the relation $p_t(T) = \exp\{-(T-t)r_t(T)\}$. It follows from (31)–(33) that

$$r_t(T) = \frac{1 - \exp\{-a\Delta\}}{a\Delta} r_t + \left(b - \frac{\sigma_r^2}{2a^2}\right) \left(1 - \frac{H(\Delta)}{\Delta}\right) + \frac{\sigma_r^2}{4a} \frac{H^2(\Delta)}{\Delta}, \quad (34)$$

where $\Delta = T - t$. Since $H(\Delta)$ is uniformly bounded, it follows from (34) that all yield curves have the same asymptote

$$r_t(\infty) = \lim_{T \rightarrow \infty} r_t(T) = b - \frac{\sigma_r^2}{2a^2}. \quad (35)$$

It is easy to show that this is also the asymptotic value of the forward rate, $f_t(T)$, defined by $f_t(T) = -\partial \log p_t(T) / \partial T$.

⁹Formally, the pricing formula (31) follows from (30) by a no-arbitrage argument under the additional assumption that the *market price of (interest rate) risk* is an affine function of r_t . The chosen parametrization corresponds to the market price of risk being of the form $\lambda_t = \{(a - \kappa)r_t + \kappa\bar{r} - ab\} / \sigma_r$. The price of a zero-coupon bond can be obtained by the risk-neutral valuation formula (1), where the risk-neutral interest rate dynamics are given by $dr_t = a(b - r_t)dt + \sigma_r dW_t^Q$.

Returning to (34), we see that a determines how strongly the short rate influences yields at longer maturities and thereby the curve steepness. Large values of a imply fast convergence to the asymptotic value and hence steep yield curves, while low values of a imply slow convergence and hence flat yield curves.

In order to evaluate the effect of inflation we introduce a price index process, I , which measures the nominal price of a real consumption good. We assume that I starts at 1 and that it subsequently evolves according to the following dynamics

$$\frac{dI_t}{I_t} = \pi_t dt + \sigma_I dW_t^I \quad (36)$$

and

$$d\pi_t = \beta(\bar{\pi} - \pi_t)dt + \sigma_\pi dW_t^\pi, \quad (37)$$

where π_t is the expected rate of inflation, $\bar{\pi}$ is the long-run mean inflation rate, β describes the degree of mean reversion, σ_π is the volatility of the expected inflation rate, while σ_I is the volatility of the price index. Thus the price index is influenced by both expected inflation and unexpected inflation shocks, where the expected inflation forms a persistent process while the shocks are mutually independent.

We will present an equilibrium analysis and for that reason we will be using long-run consensus parameter values rather than historical estimates. We let $\bar{r} = 3\%$ and $\bar{\pi} = 2\%$. This corresponds (approximately) to the inflation target of the European Central Bank (ECB) and a long-run real interest rate of 1%. We assume that both the short rate and the expected inflation are persistent processes, which over prolonged periods can deviate substantially from the long-run mean. Hence we use a low degree of mean reversion ($\kappa = \beta = 5\%$) and moderate volatility ($\sigma_r = \sigma_\pi = 0.5\%$).

The steepness of the yield curve determines the size of the term premium earned by long guarantees. We let the asymptotic interest (and forward) rate equal 4.2%, which corresponds to the ultimate forward rate (UFR) of the Solvency 2 regulatory framework. However, since we do not wish to exaggerate the term premium we assume a very slow convergence towards this level, i.e. very flat yield curves. Specifically, we use $a = 3\%$ and $b = 5.6\%$. With a short rate of e.g. $r_t = 3\%$ this leads to a modest market price of interest risk of 16% ($= -\{(a - \kappa)3\% + \kappa\bar{r} - ab\}/\sigma_r$).

The inflation tracking properties of short guarantees depend crucially on the correlation between inflation and (short) interest rates. We let ρ denote the correlation between the two underlying driving Brownian motions W^r and W^π . In the analysis we vary ρ from 0 to 1 to illustrate the sensitivity to this assumption. Finally, we assume that there is a modest amount of unexpected inflation ($\sigma_I = 0.25\%$) and that unexpected inflation is independent of the other stochastic factors, $W^I \perp\!\!\!\perp (W^r, W^\pi)$. Note that this assumptions implies that even if $\rho = 1$, the short rate will not be perfectly correlated with (increments of) the price index.

6.2 Simulation

We consider the same basic setup as in Section 5. We assume that age of retirement is $P = 65$ years and that contributions are paid annually from age 25 to age 64, i.e. 40 contributions in total. For ease of notation we assume that the person under

consideration is born at time $b = 0$. We disregard longevity risk and use the mortality law given by (21) of Section 4 for all t , i.e. $\mu_{t,b} = \mu_b$.

In practise contributions to pension schemes are often calculated as a percentage of the wage. Contributions linked to (wage) inflation offer partial protection of the purchase power of the final pension, since if inflation increases during the accumulation phase future contributions will be higher. To separate this effect from the inflation protection properties of the product itself we will consider two different contribution streams. In both cases we assume an initial (guarantee) contribution of $g_{25} = 100$. In the first case, we assume that future contributions increase with the average expected inflation of $\bar{\pi} = 2\%$, i.e. $g_t = 100 \exp\{(t - 25)2\%\}$ for $t = 25, \dots, 64$. In the second case, we assume that contributions are price indexed, i.e. $g_t = 100I_t/I_{25}$ for $t = 25, \dots, 64$.

We consider rolling annuities with guarantee periods of $L = 1, 2, \dots, 30$ years under various assumptions of the correlation, ρ , between (expected) inflation and the short rate. For each combination of L , ρ and contribution indexation we calculate the level of the final annuity in real terms, $z(P)/I_P$, in 100.000 scenarios. All scenarios are started at the long-run mean, $r_{25} = 3\%$, $\pi_{25} = 2\%$ and $I_{25} = 1$.

As described in Section 3.1, the member gets a minimum guarantee at the time of contribution and a series of subsequent guaranteed increases, all of which are calculated at prevailing market rates. The (inverse of) the increment factor at time τ is denoted ξ_τ and is given by formula (6). The factor of the last increment prior to retirement is given by an integral of the zero-coupon bond functional and the survival function over the retirement period. This representation is useful for theoretical considerations, but it is costly in a simulation context since it has to be evaluated numerically.¹⁰ For reasons of computational efficiency we therefore replace the integral with a sum from $P = 65$ years to an assumed maximum age of 120 years; tantamount to replacing the original continuous benefit stream with benefits being paid once a year at the beneficiary's birthday as long as she is alive.

For each scenario we need to simulate $\{(r_t, I_t) : t = 26, \dots, 65\}$. First note, that by Itô's lemma we have for $\delta \geq 0$ the integral representations

$$r_{t+\delta} = \bar{r} + e^{-\kappa\delta}(r_t - \bar{r}) + \sigma_r \int_t^{t+\delta} e^{-\kappa(t+\delta-s)} dW_s^r, \quad (38)$$

$$\pi_{t+\delta} = \bar{\pi} + e^{-\beta\delta}(\pi_t - \bar{\pi}) + \sigma_\pi \int_t^{t+\delta} e^{-\beta(t+\delta-s)} dW_s^\pi, \quad (39)$$

$$\int_t^{t+\delta} \pi_s ds = \bar{\pi}\delta + \frac{1 - e^{-\beta\delta}}{\beta}(\pi_t - \bar{\pi}) + \sigma_\pi \int_t^{t+\delta} \frac{1 - e^{-\beta(t+\delta-s)}}{\beta} dW_s^\pi, \quad (40)$$

$$I_{t+\delta} = I_t \exp \left\{ \int_t^{t+\delta} \pi_s ds - \frac{\sigma_I^2}{2} \delta + \sigma_I (W_{t+\delta}^I - W_t^I) \right\}, \quad (41)$$

where the last equality uses the assumption that W^I and W^π are independent. It follows that we can construct the path $\{(r_t, I_t) : t = 26, \dots, 65\}$ by successive simulations of $(r_{t+1}, \pi_{t+1}, \int_t^{t+1} \pi_s ds, W_{t+1}^I - W_t^I)$ given (r_t, π_t) for $t = 25, \dots, 64$. Proposition A.1 in

¹⁰In Section 4 we exploited the fact that when the yield curve is flat and the mortality law is of Gompertz-Makeham form, the integral can in fact be evaluated analytically as an "life expectancy" under a modified mortality law, cf. formula (24). This computational trick is however not available in the more general case considered here.

the appendix states the (conditional) distribution of the first three components, while the fourth component is distributed as a standard normal variable and independent of the other components (by assumption).

6.3 Results

In the following we examine how the rolling annuity performs under different lengths of the guarantee period (L), different correlations between the short interest rate and expected inflation (ρ), and different contribution indexation schemes. We consider the level of the final annuity in real terms, $z(P)/I_P$, and we take the median in this distribution as a measure of reward and the 5%-quantile as a measure of “risk”.¹¹

Figure 4 plots the median against the 5%-quantile in the case where contributions increase deterministically with 2% per year. We see that the median is increasing in the length of the guarantee, although the marginal increase is rather small after 20 years. This is consistent with the fact that the yield curves in the simulations are generally upwardly sloping, but almost flat in the long end. We also observe that the median does not depend on the assumed correlation between interest rates and inflation. Thus, regardless of the correlation longer guarantees typically perform better than shorter guarantees.

The risk, on the other hand, depends highly on the correlation between interest rates and inflation. Generally, a higher correlation implies that nominal rates possess a stronger link to real rates, which in turn implies that (nominal) guarantees become less risky in real terms, i.e. the curves shift to the right as correlation is increased. Intuitively, short guarantees should have a stronger link to inflation than longer guarantees, and consequently a lower risk in real terms. This, however, is countered by the typically higher yield of longer guarantees. Unless the correlation is very high, the latter effect dominates the former, such that longer guarantees are in fact *less* risky than shorter guarantees. Perhaps surprisingly, we see that only for correlations of about 90% or higher does the “stickiness” of long guarantees increase their risk. Even in this situation the risk is initially decreasing in the length of the guarantee, attains its minimum somewhere between 10 and 20 years, and increases only moderately for the very longest guarantees.

Figure 5 plots the median against the 5%-quantile of the real annuity level in the case where contributions are inflation indexed. This case mimics the real life situation, where contributions are often linked to wage and thereby indirectly to inflation. Compared to Figure 4 with deterministic contribution indexation, the median is essentially the same while the risk is reduced markedly. The inflation indexation of contributions effectively mitigates the inflation risk of longer guarantees such that longer guarantees are now less risky than shorter guarantees for all correlations except the most extreme.

We finally note, that the kinks in the curve for $\rho = 1$ coincide with changes in the maximal number of guaranteed increases of the rolling annuity. For guarantees of 20 years or more there is at most one subsequent increase, for guarantees between 14 and 19 years there are at most two subsequent increases, for guarantees between 10 and 13 years there are at most three subsequent increases and so forth. The kinks are also visible in the other curves, but much less pronounced.

¹¹Note that, since higher quantiles are less risky than lower quantiles, the 5%-quantile is a measure of lack of risk rather than of risk. For this reason the “risk-reward” plots of Figures 4 and 5 are plotted with the abscissa inverted.

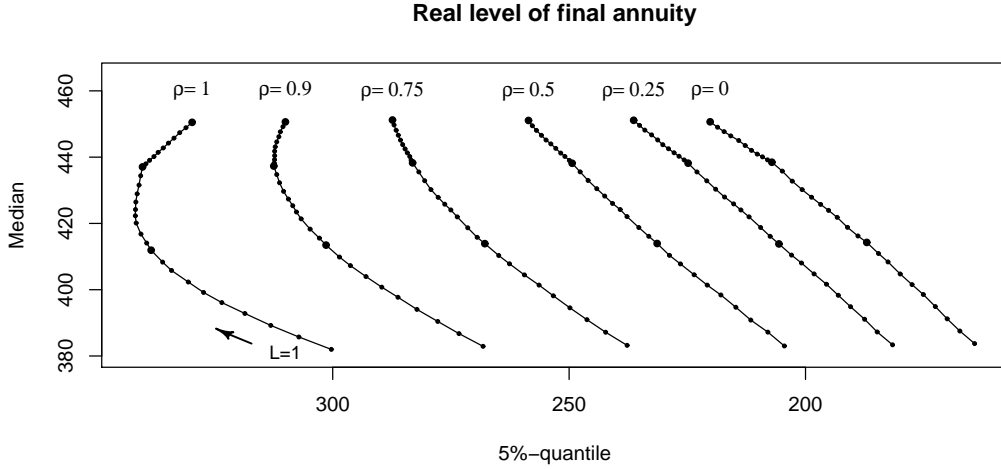


Figure 4: Plot of the median against the 5%-quantile of the final annuity level in real terms, $z(P)/I_P$, for varying correlation, ρ , between interest rates and inflation. For each correlation the guarantee period, L , is varied from 1 year (lower dots) to 30 years (upper dots). Guarantee periods of 10, 20 and 30 years are marked with larger dots for visual aid. Contributions are indexed with 2% each year.

In economics, the standard approach to comparing (and optimizing) payout profiles is through their expected utility. We consider as before the level of the final annuity in real terms and measure the utility by a constant relative risk aversion (CRRA) function,

$$E \left[u \left(\frac{z(P)}{I_P} \right) \right] = E \left[\frac{1}{1-\gamma} \left(\frac{z(P)}{I_P} \right)^{1-\gamma} \right], \quad (42)$$

where $\gamma > 0$ is a risk aversion parameter.¹² In contrast to the previous analysis where we looked at two specific quantiles in the distribution, the utility approach takes the entire distribution into account.

The *certainty equivalent* (CE) is defined as the constant (real) payout that yields the same utility as a given stochastic payout. It follows from (42) that with CRRA utility the certainty equivalent is given by

$$CE = \left[E \left(\frac{z(P)}{I_P} \right)^{1-\gamma} \right]^{1/(1-\gamma)}. \quad (43)$$

The certainty equivalent can be interpreted as the fixed (real) payout you are willing to accept in exchange for the original stochastic payout. This interpretation yields the certainty equivalent a more appealing measure than utility itself which is purely an ordinal measure for ranking different profiles.

Figure 6 plots the certainty equivalent as a function of the length of the guarantee in the case where contributions are indexed with 2% per year. The four plots show the certainty equivalent for risk aversion parameters ranging from low risk aversion ($\gamma = 2$) to very high risk aversion ($\gamma = 20$). In each plot the certainty equivalent is computed for moderate ($\rho = 0.5$) to full ($\rho = 1$) correlation between interest rates and inflation.

¹²By a limit argument, the special case $\gamma = 1$ corresponds to $u(x) = \log(x)$.

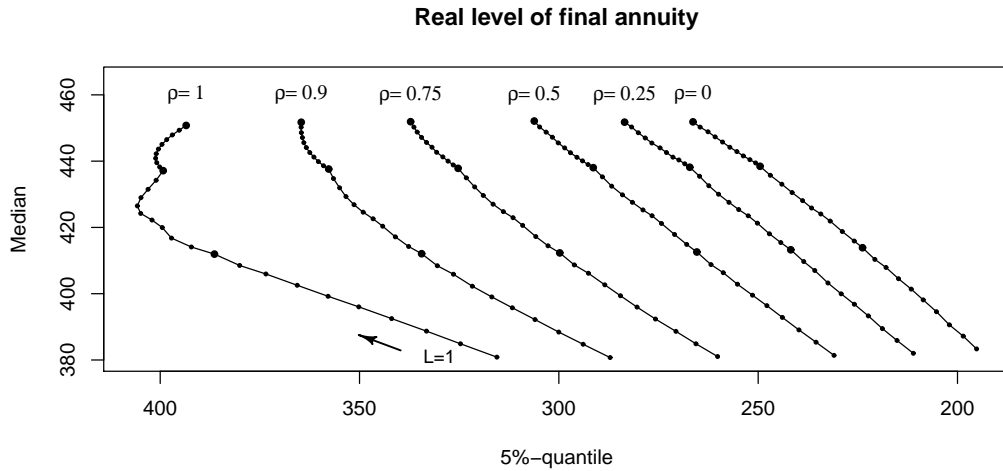


Figure 5: Plot of the median against the 5%-quantile of the final annuity level in real terms, $z(P)/I_P$, for varying correlation, ρ , between interest rates and inflation. For each correlation the guarantee period, L , is varied from 1 year (lower dots) to 30 years (upper dots). Guarantee periods of 10, 20 and 30 years are marked with larger dots for visual aid. Contributions are inflation indexed.

For low risk aversion the certainty equivalent is only slightly smaller than the median payout (shown in Figure 4), and it depends only little on the correlation. In this case you are not overly concerned with adverse outcomes and you demand a fixed payment close to the typical outcome. Since longer guarantees typically have a higher yield than shorter guarantees you prefer the longest possible guarantee.

Three things happen as the level of risk aversion increases. First, you are willing to accept smaller and smaller fixed (real) payments as you become more and more adverse to bad outcomes, i.e. the certainty equivalent decreases. Second, the certainty equivalent depends more and more on the correlation, i.e. the gap between the curves widens. With a lower correlation the fraction of adverse outcomes increases and you are willing to accept a smaller and smaller compensation to avoid the increasing number of bad outcomes. With higher risk aversion this effect becomes stronger. Third, when correlation is high you prefer shorter guarantees, i.e. the upper curves in the lower plots become hump-shaped. Longer guarantees typically perform better than shorter guarantees, but when correlation is high shorter guarantees have the benefit of fewer bad outcomes. When your risk aversion increases you value the inflation protection property of short guarantees more and more.

In summary, we find that the higher yield of longer guarantees generally outweighs the closer link to inflation of shorter guarantees. Shorter guarantee are preferred only when correlation and risk aversion are both high. In this situation, the optimal length of the guarantee is 10 to 20 years.

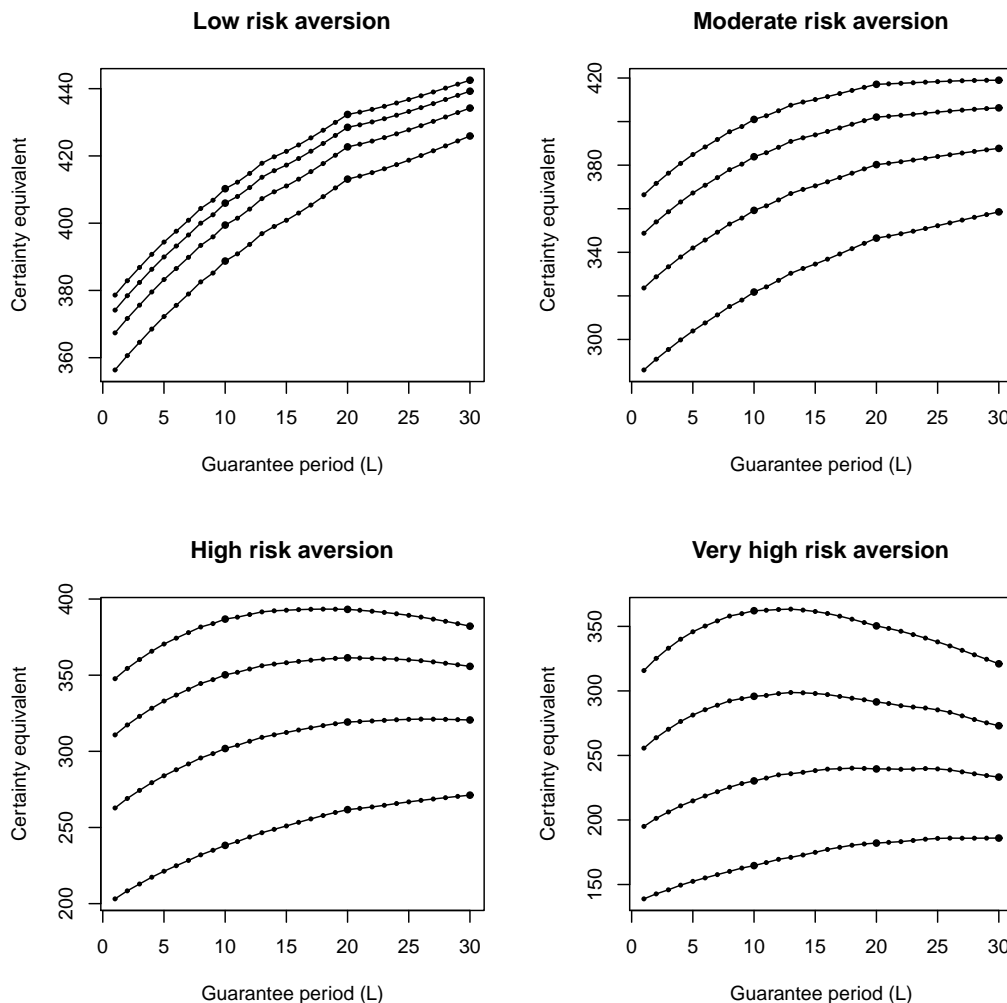


Figure 6: Plots of the certainty equivalent of the real annuity level as a function of the guarantee period. Contributions are indexed with 2% each year. From lower to upper, the curves in each plot have correlations, ρ , between interest rates and inflation of 0.5, 0.75, 0.9 and 1, respectively. The different degrees of risk aversion correspond to $\gamma = 2$ (low), $\gamma = 5$ (moderate), $\gamma = 10$ (high) and $\gamma = 20$ (very high).

6.4 Comments

The real value of the rolling annuity payout depends on the slope of the yield curve (term premium) and the correlation between interest rates and inflation. Even with modest term premiums (as in our model) and high correlation, we generally find that longer guarantees are preferable to shorter guarantees.

The presented analysis quantifies the risk-reward trade-off in equilibrium. It might be argued that with current interest rates at historically low levels, long guarantees should not be issued. The current market situation certainly speaks in favor of shorter guarantees. However, even a small term premium might be sufficient to merit longer guarantees unless increased interest rates are imminent.

7 Conclusion

In this paper we have proposed a new type of with-profits annuities: the rolling annuity. We have provided the tariff, derived the reserve and shown financial fairness. We have also considered longevity risk, the duration of the liabilities, and the real value of the payout via a simulation study.

The rolling annuity guarantees a lifelong benefit in the form of an initial minimum guarantee and a series of subsequent guaranteed increases prior to retirement. The key insight is that, prior to the final increase, the liability can be hedged by a zero-coupon bond maturing at the time for the next increase. This implies that the rolling annuity can be financially hedged at large scale as long as the guarantee period does not extend beyond the horizon of the liquid part of the relevant interest rate market, say 20 years. Keeping the guarantee period below 20 years also implies that the financial value and the regulatory value under Solvency 2 will be very similar, which simplifies risk management considerably. The rolling annuity implemented at ATP has a guarantee period of 15 years.

In the rolling annuity as here presented, “life expectancy assumptions are guaranteed”. This means that life expectancy increases beyond what was assumed at the time of contribution have no (adverse) effect on the final annuity level. This is an attractive guarantee from the members’ point of view, but it entails longevity risk for the provider. At ATP, members acquire rolling annuities for only 80 pct. of their contributions, while the remaining 20 pct. enter collective free funds. The free funds act as risk capital for covering longevity and other risks. Pension funds with different ownership and capital structures and facing different regulatory requirements are likely to provide risk capital by other means. Alternatively, longevity risk and thereby the need for risk capital can be reduced by weakening the “life expectancy guarantee”, e.g. by reducing the subsequent increases if life expectancy evolves faster than expected.

The rolling annuity is intended to form part of a with-profits contract, where the rolling annuity provides the guarantees and a return-seeking portfolio provides excess returns via exposure to e.g. the stock market. At ATP the free funds provide risk capital to a substantial portfolio harvesting a variety of market risk factors. The value created from these activities are transferred to the members in the form of additional annuity indexation. Other pension funds might prefer different profit-sharing mechanisms.

The ultimate purpose of a pension is to provide an income stream in retirement. Protecting the purchasing power of the pension should be a key objective of the product design. The simulation study shows that the guarantees themselves, and in particular in combination with inflation indexed contributions, offer some but far from perfect inflation protection. Additional inflation protection can be achieved by including inflation exposure in the return-seeking portfolio, either indirectly via e.g. commodities and real estate, or directly via index bonds, inflation swaps or other derivatives.

A Auxiliary distributional result

The simulation of the capital market in Section 6 is performed by use of the following distributional result, which we state without proof. The result is partly contained in Theorem 1 of Ben-Ameur et al. (2007); the full result can be proved along the same lines as their result.

Proposition A.1. *Let the dynamics of r_t and π_t be given by (30) and (37) of Section 6.1, and let ρ denote the correlation between W^r and W^π .*

For any t and any $\delta \geq 0$ the conditional distribution of $(r_{t+\delta}, \pi_{t+\delta}, \int_t^{t+\delta} \pi_s ds)'$ given (r_t, π_t) is multivariate normal,

$$\left(r_{t+\delta}, \pi_{t+\delta}, \int_t^{t+\delta} \pi_s ds \right)' \mid (r_t, \pi_t) \sim N_3(m(\delta, r_t, \pi_t), \Sigma(\delta)), \quad (44)$$

with mean vector and covariance matrix given by

$$m(\delta, r_t, \pi_t) = \begin{pmatrix} \bar{r} + e^{-\kappa\delta}(r_t - \bar{r}) \\ \bar{\pi} + e^{-\beta\delta}(\pi_t - \bar{\pi}) \\ \bar{\pi}\delta + \frac{1-e^{-\beta\delta}}{\beta}(\pi_t - \bar{\pi}) \end{pmatrix}, \quad \Sigma(\delta) = \begin{pmatrix} \Sigma_{11}(\delta) & \Sigma_{12}(\delta) & \Sigma_{13}(\delta) \\ \Sigma_{21}(\delta) & \Sigma_{22}(\delta) & \Sigma_{23}(\delta) \\ \Sigma_{31}(\delta) & \Sigma_{32}(\delta) & \Sigma_{33}(\delta) \end{pmatrix},$$

where

$$\begin{aligned} \Sigma_{11}(\delta) &= \sigma_r^2 \frac{1 - e^{-2\kappa\delta}}{2\kappa}, \\ \Sigma_{22}(\delta) &= \sigma_\pi^2 \frac{1 - e^{-2\beta\delta}}{2\beta}, \\ \Sigma_{33}(\delta) &= \frac{\sigma_\pi^2}{2\beta^3} (-3 + 2\beta\delta + 4e^{-\beta\delta} - e^{-2\beta\delta}), \\ \Sigma_{12}(\delta) = \Sigma_{21}(\delta) &= \rho\sigma_r\sigma_\pi \frac{1 - e^{-(\beta+\kappa)\delta}}{\beta + \kappa}, \\ \Sigma_{13}(\delta) = \Sigma_{31}(\delta) &= \frac{\rho\sigma_r\sigma_\pi}{\beta} \left(\frac{1 - e^{-\kappa\delta}}{\kappa} - \frac{1 - e^{-(\beta+\kappa)\delta}}{\beta + \kappa} \right), \\ \Sigma_{23}(\delta) = \Sigma_{32}(\delta) &= \frac{\sigma_\pi^2}{2\beta^2} (1 - 2e^{-\beta\delta} + e^{-2\beta\delta}). \end{aligned}$$

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