Inconsistent investment and consumption problems

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Abstract

In a traditional Black-Scholes market we develop a verification theorem for a general class of investment and consumption problems where the standard dynamic programming principle does not hold. The theorem is an extension of the standard Hamilton-Jacobi-Bellman equation in the form of a system of non-linear differential equations. We derive the optimal investment and consumption strategy for a mean-variance investor without pre-commitment endowed with labor income. In the case of constant risk aversion it turns out that the optimal amount of money to invest in stocks is independent of wealth. The optimal consumption strategy is given as a deterministic bang-bang strategy. In order to have a more realistic model we allow the risk aversion to be time and state dependent. Of special interest is the case were the risk aversion is inversely proportional to present wealth plus the financial value of future labor income net of consumption. Using the verification theorem we give a detailed analysis of this problem. It turns out that the optimal amount of money to invest in stocks is given by a linear function of wealth plus the financial value of future labor income net of consumption. The optimal consumption strategy is again given as a deterministic bang-bang strategy. We also calculate, for a general time and state dependent risk aversion function, the optimal investment and consumption strategy for a mean-standard deviation investor without pre-commitment. In that case, it turns out that it is optimal to take no risk at all.

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1 Introduction

The dynamic asset allocation problem for a portfolio investor searching to maximize the mean-variance objective

$$E_{t,x}\left[X^{\pi}(T)\right] - \frac{\gamma}{2} Var_{t,x}\left[X^{\pi}(T)\right], \tag{1}$$

for a constant γ, has in recent years been subject to numerous studies. The problem is non-standard in the sense that it cannot be formalized as a standard stochastic control problem,

$$\sup_{u} E_{t,x}\left[\int_t^T C(s, X^u(s), u(s))ds + \phi(X^u(T))\right], \tag{2}$$

for some functions C and ϕ. Therefore, the traditional dynamic programming approach does not apply directly. This is due to the lack of the iterated expectation property, and, consequently, we refer to such problems as *time inconsistent*. For every time inconsistent control problem we can fix an initial point and then solve the problem. The corresponding optimal control will at a later fixed point in time then not be optimal. We refer to this solution as the optimal *pre-commitment* control (for the mean-variance case see Korn (1997) and Zhou and Li (2000)). The first to solve the problem (1) *without pre-commitment* were Basak and Chabakauri (2010). They present the problem in a quite general incomplete Wiener driven framework and by applying a so called total variance formula they obtain an extension of the classical Hamilton-Jacobi-Bellman equation for solving the problem. Björk and Murgoci (2009) extend the class of standard solvable problems (2) to the class of objectives

$$E_{t,x}\left[\int_t^T C(x, s, X^u(s), u(s))ds + \phi(x, X^u(T))\right] + G\big(x, E_{t,x}\left[X^u(T)\right]\big), \tag{3}$$

for some function G. They work in a general Markovian financial market having the results of Basak and Chabakauri (2010) as a special case. In (3) time inconsistency enters at two points: First, the present state x appears in C, ϕ and G, and second, the function G is allowed to be non-linear in the conditional expectation. Another work having Basak and Chabakauri (2010) as a special case is Kryger and Steffensen (2010). They analyze, in a classic Black-Scholes market, the class of problems given by the objectives

$$f\left(t, x, E_{t,x}\left[\phi_1(X^{\pi}(T))\right], \ldots, E\left[\phi_n(X^{\pi}(T))\right]\right), \tag{4}$$

where f is allowed to be a non-affine function of the expectation of the ϕ functions. One special example of interest only contained in (4) is the dynamic asset allocation problem for a portfolio investor with mean-standard deviation criteria. Kryger and Steffensen (2010) show that the optimal strategy derived for a mean-standard deviation investor without pre-commitment is to take no risk at all. The latest contribution to the literature treating mean-variance optimization problems without pre-commitment comes from Björk et al. (2012). They argue that the somehow unsatisfactory solution to (1), saying that the optimal amount to invest in stocks is constant, is due to the fact that the risk aversion parameter γ is constant. They solve the problem (1) for a general risk aversion function $\gamma(x)$ depending on present wealth and obtain for the special case $\gamma(x) = \gamma/x$ that the corresponding optimal amount invested in stocks is linear in wealth.

Working with inconsistent stochastic optimization problems without pre-commitment it might not be totally clear what we mean by an optimal control. This is well discussed in both Björk and Murgoci (2009) and Björk et al. (2012). They argue that the right thing to do is to study time inconsistency within a game theoretic framework and then look for a subgame perfect Nash equilibrium point for this game. This approach is first described in Strotz (1955), and the first to give a precise definition of the game theoretic equilibrium concept in continuous time were

Ekeland and Lazrak (2006), and, Ekeland and Pirvu (2008). Conceptually, we attack the problems in the same manner.

Björk and Murgoci (2009) also take consumption into account. However, their preferences over consumption do not contribute to the inconsistency in the sense that the consumption term is just added as in a standard stochastic control problem, see (3) and (2). In this paper we introduce a new class of optimization problems. In these problems inconsistency also arises from taking a non-linear function of the expected (utility of) consumption. In addition we also allow for a capital injection in the form of a deterministic labor income. This leads to some mathematical difficulties but we manage to establish a verification theorem containing a Bellman-type set of differential equations for determination of the optimal strategies. Two concrete examples of economic interest covered by our approach are the mean-variance and the mean-standard deviation problems without pre-commitment including consumption and labor income. Those cases are analyzed in details in Section 3–5. One should recognize that we consider the problems for a general risk aversion function. More specific, we allow the risk aversion function to be both time and state dependent.

The structure of the paper is as follows: In Section 2 we present our formal model and the problems of interest. We discuss the concept of inconsistency, admissible strategies, and what we mean by an optimal strategy. Finally a verification theorem characterizing the solution to our class of problems is provided, leaving the proof to the Appendix. In Section 3 we derive the optimal consumption and investment strategy for a mean-variance investor without pre-commitment and constant risk aversion. In Section 4 we derive the optimal consumption and investment strategy for a mean-variance investor without pre-commitment and risk aversion inversely proportional to present wealth plus the financial value of future labor income net of consumption. In Section 5 we show, for a general time and state dependent risk aversion function fulfilling some reasonable assumptions, that a mean-standard deviation investor without pre-commitment should optimally take no risk at all.

2 The basic framework

In this section we present the basic model and the problems of interest. We discuss what we mean by an inconsistent problem, admissible strategies, and a corresponding optimal strategy. To solve the problems one need a Bellman-type set of partial differential equations. These are presented at the end of this section in the verification theorem, Theorem 2.1. Proofs are outlined in the Appendix.

2.1 The economic model

The economic setup is a standard Black-Scholes model consisting of a bank account, B, with risk free short rate, r, and a stock, S, with dynamics given by

$$dB(t) = rB(t)dt,\ B(0) = 1,$$
$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t),\ S(0) = s_0 > 0.$$

Here $\alpha, \sigma, r > 0$ are constants and it is assumed that $\alpha > r$. The process W is a standard Brownian motion on an abstract probability space $(\Omega, \mathcal{F}, P)$ equipped with the filtration $\mathbb{F}^W = \left(\mathcal{F}^W(t)\right)_{t\in[0,T]}$ given by the P-augmentation of the filtration $\sigma\{W(s);\, 0 \le s \le t\}$, $\forall t \in [0,T]$.

We consider an investor with time horizon $[0,T]$, $T > 0$, and wealth process $(X(t))_{t\in[0,T]}$. The investor is assumed to be endowed with a continuous deterministic labor income rate ℓ and an initial amount of money x_0. At time t the investor chooses a non-negative consumption rate $c(t)$ and places a proportion $\pi(t)$ of his wealth in the stock, and the remainder in the bank account. Denoting by $X^{c,\pi}(t)$ the investors wealth at time t given the consumption and investment strategy (c,π), from now on just called strategy, the dynamics of the investor's wealth become

$$dX^{c,\pi}(t) = [(r+\pi(t)(\alpha-r))X^{c,\pi}(t)+\ell(t)-c(t)]\,dt+\pi(t)\sigma X^{c,\pi}(t)dW(t),\ t\in[0,T), \qquad (5)$$
$$X(0)=x_0>0.$$

For later use, see Section 4, we present the equivalent martingale measure, $\widehat{P}$, which for the Black-Scholes market is well-known to be given by the unique Radon-Nikodym derivative

$$\frac{d\widehat{P}(t)}{dP(t)} = \exp\left(-\left(\frac{\alpha-r}{\sigma}\right)W(t)-\frac{1}{2}\left(\frac{\alpha-r}{\sigma}\right)^2 t\right),\ t\in[0,T]. \qquad (6)$$

The process $W^{\widehat{P}}$ given by

$$W^{\widehat{P}}(t)=W(t)+\frac{\alpha-r}{\sigma}t,\ t\in[0,T],$$

is a standard Brownian motion under the martingale measure $\widehat{P}$.

2.2 The problems of interest

Before introducing the problems of interest we introduce two conditional expectations

$$y^{c,\pi}(t,x)=E\left[\int_t^T e^{-\rho(s-t)}c(s)ds+e^{-\rho(T-t)}X^{c,\pi}(T)\,\middle|\,X(t)=x\right],$$
$$z^{c,\pi}(t,x)=E\left[\left(\int_t^T e^{-\rho(s-t)}c(s)ds+e^{-\rho(T-t)}X^{c,\pi}(T)\right)^2\,\middle|\,X(t)=x\right].$$

Here ρ is a constant discounting rate, possibly different from the interest rate r. Loosely speaking, the class of stochastic problems we consider is, for any $(t,x)\in[0,T)\times\mathbb{R}$, to maximize

$$f^{c,\pi}(t,x,y^{c,\pi}(t,x),z^{c,\pi}(t,x)),\ (c,\pi)\in\mathcal{A}, \qquad (7)$$

where $f\in\mathcal{C}^{1,2,2,2}$ and $\mathcal{A}$ is the class of admissible strategies to be defined in Theorem 2.1. The class of problems given by (7) contains two examples (among others) of economic interest, which we analyze in Section 3–5:

- Mean-variance without pre-commitment:

$$f^{c,\pi}(t,x,y,z)=y-\frac{\psi(t,x)}{2}\left(z-y^2\right),\ (t,x)\in[0,T)\times\mathbb{R}, \qquad (8)$$

  where $\psi\in\mathcal{C}^{1,2}$ is a function which is allowed to depend on time and wealth. The problem is non-standard because of the non-linearity in y and because of the presence of t and x in the function ψ[1]. For a pure portfolio investor (c and ℓ set to zero) variants of (8) have been treated: For ψ constant, the problem is treated, in an incomplete Wiener driven framework, by Basak and Chabakauri (2010). Further, the problem is studied as a special (the simplest) case by Björk and Murgoci (2009). The case $\psi(t,x)=\gamma/x$, for a constant γ, is investigated by Björk et al. (2012).

[1]The appearance of z and thereby terms like "the conditional expectations of cumulated consumption to the power of 2" also makes the problem non-standard.

- Mean-standard deviation without pre-commitment:

$$f^{c,\pi}(t,x,y,z) = y - \psi(t,x)\left(z-y^2\right)^{\frac{1}{2}}, \ (t,x) \in [0,T) \times \mathbb{R}, \tag{9}$$

  where $\psi \in \mathcal{C}^{1,2}$ is a function allowed to depend on time and wealth. In addition to the arguments for the mean-variance problem this problem is non-standard due to the non-linearity in z. For a pure portfolio investor (c and ℓ set to zero) the problem has been treated in Kryger and Steffensen (2010) for the case ψ constant.

To the authors knowledge mean-variance and mean-standard deviation without pre-commitment including consumption and terminal wealth have not before been analyzed.

2.3 Inconsistency and the concept of an optimal strategy

The stochastic problems presented in (8) and (9) are called *time inconsistent* in the sense that the Bellman Optimality Principle does not hold: Suppose that we find the optimal strategy (c^*,π^*) for the time-0 problem $\mathcal{P}_{0,x_0}$ and suppose that we use this strategy on the time interval $[0,t]$. Then at time t the strategy (c^*,π^*) will not be optimal for the time-t problem $\mathcal{P}_{t,X^{c^*,\pi^*}(t)}$. This is because the law of iterated expectations does not apply for a given strategy. If the investors preference really is to pre-commit at time 0, he should of course simply solve the problem $\mathcal{P}_{0,x_0}$ and follow the corresponding optimal pre-commitment strategy. Here optimal is interpreted as optimal from the point of view of time zero. For some investors, this might be meaningful.

On the other hand it could easily be argued that most investors assign the same weight to all points in time, i.e. they do not look for an optimal strategy from the point of view of (say) time zero. Or put it another way, it seems reasonable that the investor assign no particular importance to a single point in time. Therefore, Björk and Murgoci (2009) and Björk et al. (2012) attack the problems in a game theoretic framework. That is, our preferences change in a temporally inconsistent way as time goes by and we can thus think about the problem as a game where the players are the future incarnations of ourselves. More specific, at every point in time t we have a player (and incarnation of ourselves) which we denote $\mathcal{P}_t$. Player $\mathcal{P}_t$ chooses the strategy $(c(t),\pi(t))$ at time t. The reward to $\mathcal{P}_t$ of course depends on the choice made by $\mathcal{P}_t$, but also on the choices made by the players $\mathcal{P}_s$ for all $s \in (t,T]$. We can now loosely define a subgame perfect Nash equilibrium strategy as a strategy (c^*,π^*) for which the following holds (for all players):

- If $\mathcal{P}_t$ knows that all players coming after him will use the strategy (c^*,π^*), then it is optimal for $\mathcal{P}_t$ also to use (c^*,π^*).

In a discrete time setup the concept of a subgame perfect Nash equilibrium strategy is very intuitive and it seems natural to look for subgame perfect Nash equilibrium strategies. However, it turns out to be a lot more complicated to define an equilibrium strategy in continuous time. The problem is that a single point in time has Lebesque measure zero, i.e. the individual player $\mathcal{P}_t$ does not influence the outcome of the game. Ekeland and Lazrak (2006) and Ekeland and Pirvu (2008) gave a precise definition of the game theoretic equilibrium concept in continuous time:

Definition 2.1. *Consider a strategy (c^*,π^*), choose a fixed reel point (c,π), a fixed real number $h>0$, and an arbitrary initial point (t,x). Define the strategy $(\widetilde{c}_h,\widetilde{\pi}_h)$ by*

$$(\widetilde{c}_h(s),\widetilde{\pi}_h(s)) = \begin{cases} (c,\pi), & \text{for } t \le s < t+h, \\ (c^*(s),\pi^*(s)), & \text{for } t+h \le s < T. \end{cases} \tag{10}$$

If

$$\liminf_{h\to 0} \frac{f^{c^*,\pi^*}\left(t,x,y^{c^*,\pi^*}(t,x),z^{c^*,\pi^*}(t,x)\right) - f^{\widetilde{c}_h,\widetilde{\pi}_h}\left(t,x,y^{\widetilde{c}_h,\widetilde{\pi}_h}(t,x),z^{\widetilde{c}_h,\widetilde{\pi}_h}(t,x)\right)}{h} \ge 0,$$

for all $(c,\pi) \in \mathbb{R}_+ \times \mathbb{R}$, we say that (c^*,π^*) is an ***equilibrium*** *strategy.*

Looking for equilibrium strategies as defined in Definition 2.1 Björk et al. (2012) solve inconsistent control problems in the form (3). Consequently, they refer to the function defining the expected value of using the equilibrium strategy as the equilibrium value function.

We also choose to look for equilibrium strategies and refer, as Björk et al. (2012), to the strategies by the term *optimal*. However, in contrast to Björk et al. (2012) we choose to refer to the corresponding value function as the *optimal* value function. Denoting the optimal value function by V we write

$$V(t,x) = f^{c,\pi}\left(t, x, y^{c^*,\pi^*}, z^{c^*,\pi^*}\right), \tag{11}$$

for a strategy (c^*, π^*) fulfilling the equilibrium criteria in Definition 2.1. Doing so, our problem is to look for the optimal value function and the corresponding optimal strategies for objectives in the form given by (7).

Remark 2.1. It is all about how we look at the problem. Assume we are standing at time t and consider, for example, the mean-variance problem given by

$$f^{c^*,\pi^*}(t, x, y^{c,\pi}(t,x), z^{c,\pi}(t,x)), \tag{12}$$

where $f^{c^*,\pi^*}(t,x,y,z) = y - \frac{\gamma}{2}\left(z - y^2\right)$. To write the pre-commitment version of this problem (that is the investor pre-commit to his time-t preferences) one should define

$$\widetilde{y} = E\left[\int_t^T e^{-\rho(s-t)}c(s)ds + e^{-\rho(T-t)}X^{c,\pi}(T) \,\middle|\, X(t) = x\right],$$

and then write the problem as

$$\widetilde{f}^{c^*,\pi^*}(t, x, y^{c,\pi}(t,x), z^{c,\pi}(t,x), \widetilde{y}), \tag{13}$$

where $\widetilde{f}^{c^*,\pi^*}(t,x,y,z,\widetilde{y}) = y - \frac{\gamma}{2}\left(z + (\widetilde{y})^2 - 2y\widetilde{y}\right)$. At this point the reader might be confused since, at time t, we obviously have

$$f^{c^*,\pi^*}(t, x, y^{c,\pi}(t,x), z^{c,\pi}(t,x)) = \widetilde{f}^{c^*,\pi^*}(t, x, y^{c,\pi}(t,x), z^{c,\pi}(t,x), \widetilde{y}),$$

i.e. from the starting point of view the two problems look identical. However, at time $s \in (t,T)$, we have

$$f^{c^*,\pi^*}(s, X^{c,\pi}(s), y(s, X^{c,\pi}(s)), z(s, X^{c,\pi}(s))) \neq \widetilde{f}^{c^*,\pi^*}(s, X^{c,\pi}(s), y(s, X^{c,\pi}(s)), z(s, X^{c,\pi}(s)), \widetilde{y}).$$

The point is that by (13) we have made it clear that $\widetilde{y}$, opposed to x, y and z, is not a dynamic variable, i.e. the investor pre-commits to the time-t target when evaluating the variance term. To solve the pre-commitment problem (13) the trick is to write the problem as

$$\sup_{(c,\pi)\in\mathcal{A},\ \widetilde{y}=K} \widetilde{f}^{c^*,\pi^*}(t, x, y^{c,\pi}(t,x), z^{c,\pi}(t,x), K).$$

This can be solved in two steps: First solve the problem for a general K (this is a standard control problem), thereby obtaining an optimal strategy, $(c^*(K), \pi^*(K))$, as a function of K. Then insert $(c^*(K), \pi^*(K))$ in $\widetilde{y}$ and determine the optimal K^* as the solution to the nonlinear equation $\widetilde{y} = K^*$. For references see Korn (1997) and Zhou and Li (2000).

2.4 The main result

In this subsection we present an extension of the standard Hamilton-Jacobi-Bellman (HJB) equation for characterization of the optimal value function and the corresponding optimal strategy. The power of the verification theorem, as in the classic HJB framework, is that it transforms the stochastic problem into a system of deterministic differential equations and a deterministic pointwise infimum problem. The infimum equation (16) below differentiates from the standard HJB-equation by the extra terms Q, U and J. Note that the presence of the upper subscript notation[2] for the f-terms, and the presence of $F^{(1)}$ and $F^{(1)}$, in Q, U and J indicate that the (optimal) strategy fulfilling the pointwise infimum must be used at all future time points. This is in agreement with the concept of equilibrium strategies.

Theorem 2.1. *Let $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ be a function in $\mathcal{C}^{1,2,2,2}$. Define the set of admissible strategies, $\mathcal{A}$, as those, (c,π), for which the partial differential equations (63)–(64) and (70)–(71) have solutions, and for which the stochastic integrals in (69), (75) and (87) are martingales. Let, $\forall(t,x) \in [0,T) \times \mathbb{R}$, the optimal value function $V(t,x)$ be defined by (11), and define*

$$y^{c,\pi}(t,x) = E\left[\int_t^T e^{-\rho(s-t)}c(s)ds + e^{-\rho(T-t)}X^{c,\pi}(T) \,\middle|\, X(t)=x\right], \tag{14}$$

$$z^{c,\pi}(t,x) = E\left[\left(\int_t^T e^{-\rho(s-t)}c(s)ds + e^{-\rho(T-t)}X^{c,\pi}(T)\right)^2 \,\middle|\, X(t)=x\right]. \tag{15}$$

If there exist three functions F, $F^{(1)}$ and $F^{(2)}$ such that, $\forall(t,x) \in (0,T) \times \mathbb{R}$, we have

$$F_t = \inf_{(c,\pi)\in\mathcal{A}} \Big\{ -[(r+\pi(\alpha-r))x + \ell - c](F_x - Q) - \frac{1}{2}\pi^2\sigma^2x^2(F_{xx} - U) + J \Big\}, \tag{16}$$

$$F(T,x) = f\left(T,x,x,x^2\right),$$

$$F_t^{(1)} = -[(r+\pi^*(\alpha-r))x + \ell - c^*]F_x^{(1)} - \frac{1}{2}\pi^{*2}\sigma^2x^2F_{xx}^{(1)} - c^* + \rho F^{(1)}, \tag{17}$$

$$F^{(1)}(T,x) = x,$$

$$F_t^{(2)} = -[(r+\pi^*(\alpha-r))x + \ell - c^*]F_x^{(2)} - \frac{1}{2}\pi^{*2}\sigma^2x^2F_{xx}^{(2)} - 2c^*F^{(1)} + 2\rho F^{(2)}, \tag{18}$$

$$F^{(2)}(T,x) = x^2,$$

where

$$Q = f_x^{c^*,\pi^*}, \tag{19}$$

$$U = f_{xx}^{c^*,\pi^*} + \left(F_x^{(1)}\right)^2 f_{yy}^{c^*,\pi^*} + \left(F_x^{(2)}\right)^2 f_{zz}^{c^*,\pi^*} + 2F_x^{(1)}F_x^{(2)}f_{yz}^{c^*,\pi^*} + 2F_x^{(1)}f_{xy}^{c^*,\pi^*} + 2F_x^{(2)}f_{xz}^{c^*,\pi^*}, \tag{20}$$

$$J = \left(\rho F^{(1)} - c\right) f_y^{c^*,\pi^*} + 2\left(\rho F^{(2)} - cF^{(1)}\right) f_z^{c^*,\pi^*} + f_t^{c^*,\pi^*}, \tag{21}$$

and

$$(\pi^*, c^*) = \arg\inf_{(c,\pi)\in\mathcal{A}} \left\{ -[(r+\pi(\alpha-r))x + \ell - c](F_x - Q) - \frac{1}{2}\pi^2\sigma^2x^2(F_{xx} - U) + J \right\},$$

then

$$V(t,x) = F(t,x),\; y^{c^*,\pi^*}(t,x) = F^{(1)}(t,x),\; z^{c^*,\pi^*}(t,x) = F^{(2)}(t,x),$$

and the optimal strategy is given by (c^*,π^*).

Proof. See Appendix A.1. □

[2]E.g. $f_x^{c^*,\pi^*}$ which is shorthand notation for $f_x^{c^*,\pi^*}(t,x,F^{(1)}(t,x),F^{(2)}(t,x))$

3 Mean-variance with constant risk aversion

The simplest case of mean-variance optimization without pre-commitment including consumption and terminal wealth is obtained by assuming constant risk aversion. Using Theorem 2.1 we are able to derive the optimal strategy. It turns out that the optimal investment strategy corresponds to a constant amount of money invested in stocks and that the optimal consumption strategy becomes a deterministic bang-bang-strategy. The solution and the problem are discussed below.

3.1 Presenting and solving the problem

Consider the problem of finding the optimal strategy for the objective given by

$$E_{0,x_0}\left[\int_0^T e^{-\rho s}c(s)ds + e^{-\rho T}X^{c,\pi}(T)\right] - \frac{\gamma}{2}Var_{0,x_0}\left[\int_0^T e^{-\rho s}c(s)ds + e^{-\rho T}X^{c,\pi}(T)\right], \quad (22)$$

where $\gamma > 0$ is a constant defining the investor's risk aversion, and where we restrict the admissible strategies to $(c,\pi) \in \mathcal{A}\bigcap(\mathcal{D}\times\mathbb{R})$, where $\mathcal{D}(s) := [c_{\min}(s), c_{\max}(s)]$, $s \in [0,T]$, is a finite interval. The corresponding function f is given by

$$f(t,x,y,z) = y - \frac{\gamma}{2}\left(z - y^2\right).$$

The system of partial differential equations we want to solve in order to obtain the optimal value function and the optimal strategy is given by (16) and (17)[3]. A candidate for the optimal strategy in terms of the value function is found by differentiating with respect to c and π inside the curly brackets in (16). Thereby

$$c^*(t,x) = \begin{cases} c_{\max}(t), & \text{if } F_x(x,t) - Q(x,t) < 1, \\ \text{non-defined}, & \text{if } F_x(x,t) - Q(x,t) = 1, \\ c_{\min}(t), & \text{if } F_x(x,t) - Q(x,t) > 1, \end{cases} \quad (23)$$

$$\pi^*(t,x)x = -\frac{\alpha - r}{\sigma^2}\frac{F_x - Q}{F_{xx} - U}, \quad (24)$$

(provided $U > F_{xx}$). Clearly

$$\begin{aligned} &\psi_x = \psi_{xx} = \psi_t = 0, \\ &f_y = 1 + \gamma y, f_{yy} = \gamma, f_z = -\frac{\gamma}{2}, \\ &f_t = f_x = f_{xx} = f_{xz} = f_{xy} = f_{zz} = f_{yz} = 0. \end{aligned}$$

Inserting this into (19)–(21) gives

$$Q = 0, \quad (25)$$

$$U = \gamma\left(F_x^{(1)}\right)^2, \quad (26)$$

$$J = \rho F^{(1)} - c - \gamma\rho\left(F^{(2)} - \left(F^{(1)}\right)^2\right). \quad (27)$$

We now search for solutions in the form

$$\begin{aligned} F(t,x) &= A(t)x + B(t), \\ F^{(1)}(t,x) &= a(t)x + b(t), \end{aligned}$$

[3] Since $F_t = F_t^{(1)} - \frac{\gamma}{2}\left(F_t^{(2)} + 2F^{(1)}F_t^{(1)}\right)$ we only need to solve two of the three differential equations given by (16)–(18). We choose to solve (16) and (17).

where A, B, a and b are deterministic functions of time. In order to calculate J we need to derive $F^{(2)}$ from our guess. The forms of F and $F^{(1)}$ determine the form of $F^{(2)}$. We have that

$$F^{(2)}(t,x) = \frac{2}{\gamma}[a(t)x + b(t) - A(t)x - B(t)] + [a(t)x + b(t)]^2.$$

The partial derivatives of interest are

$$F_t = A'(t)x + B'(t),\ F_x = A(t),\ F_{xx} = 0,$$
$$F_t^{(1)} = a'(t)x + b'(t),\ F_x^{(1)} = a(t),\ F_{xx}^{(1)} = 0.$$

Inserting this into (25)–(27) gives

$$Q(t,x) = 0, \tag{28}$$
$$U(t,x) = \gamma a(t)^2, \tag{29}$$
$$J(t,x) = \rho[a(t)x + b(t)] - c(t) - 2\rho[a(t)x + b(t) - A(t)x - B(t)]. \tag{30}$$

Plugging the relevant derivatives, (28) and (29) into (23) and (24) we can now write the candidate for the optimal strategy in terms of the deterministic functions A and a. We get

$$c^*(t,x) = \begin{cases} c_{\max}(t), & \text{if } A(t) < 1, \\ \text{non-defined}, & \text{if } A(t) = 1, \\ c_{\min}(t), & \text{if } A(t) > 1, \end{cases} \tag{31}$$

$$\pi^*(t,x)x = \frac{1}{\gamma}\frac{\alpha - r}{\sigma^2}\frac{A(t)}{a(t)^2}, \tag{32}$$

(provided $\gamma a^2 > 0$). Inserting (28)–(32) and the relevant derivatives into the differential equations (16) and (17) and including the terminal conditions gives

$$\begin{aligned} A_t x + B_t &= -rxA - \frac{1}{2\gamma}\frac{(\alpha - r)^2}{\sigma^2}\frac{A^2}{a^2} - \ell A + c^*(A-1) \\ &\quad + \rho(ax+b) - 2\rho(ax + b - Ax - B), \\ A(T) &= 1, \\ B(T) &= 0, \\ a_t x + b_t &= -rxa - \frac{1}{\gamma}\frac{(\alpha - r)^2}{\sigma^2}\frac{A}{a} - \ell a + c^*(a-1) + \rho(ax+b), \\ a(T) &= 1, \\ b(T) &= 0. \end{aligned}$$

We obtain the solutions

$$A(t) = a(t) = e^{(r-\rho)(T-t)}$$

and

$$B(t) = e^{2\rho t}\int_t^T \left[\frac{1}{2\gamma}\frac{(\alpha - r)^2}{\sigma^2} + \ell(s)e^{(r-\rho)(T-s)} - c^*(s)\left(e^{(r-\rho)(T-s)} - 1\right) + \rho b(s)\right] e^{-2\rho s} ds, \tag{33}$$

$$b(t) = e^{\rho t}\int_t^T \left[\frac{1}{\gamma}\frac{(\alpha - r)^2}{\sigma^2} + \ell(s)e^{(r-\rho)(T-s)} - c^*(s)\left(e^{(r-\rho)(T-s)} - 1\right)\right] e^{-\rho s} ds, \tag{34}$$

as well as the relation

$$b(t) - B(t) = \frac{1}{2\gamma}\frac{(\alpha - r)^2}{\sigma^2}\int_t^T e^{-2\rho(s-t)} ds = \begin{cases} \frac{1}{4\gamma\rho}\frac{(\alpha-r)^2}{\sigma^2}\left(1 - e^{-2\rho(T-t)}\right), & \text{if } \rho > 0, \\ \frac{1}{2\gamma}\frac{(\alpha-r)^2}{\sigma^2}(T-t), & \text{if } \rho = 0. \end{cases} \tag{35}$$

The optimal strategy now follows directly from plugging the solutions for A and a into (31) and (32). We summarize the results as follows:

Proposition 3.1. *For the mean-variance problem given by (22) we have the following results.*

- *The optimal strategy is given by*

$$c^*(t,x) = \begin{cases} c_{\max}(t), & \text{if } r < \rho, \\ \text{non-defined}, & \text{if } r = \rho, \\ c_{\min}(t), & \text{if } r > \rho, \end{cases}$$

$$\pi^*(t,x)x = \frac{1}{\gamma}\frac{(\alpha - r)}{\sigma^2} e^{-(r-\rho)(T-t)}.$$

- *The optimal value function is given by*

$$V(t,x) = e^{(r-\rho)(T-t)}x + B(t).$$

- *The conditional expected value of the discounted cumulated optimal consumption plus the discounted optimal terminal wealth is given by*

$$E_{t,x}\left[\int_t^T e^{-\rho(s-t)}c^*(s)ds + e^{-\rho(T-t)}X^{c^*,\pi^*}(T)\right] = e^{(r-\rho)(T-t)}x + b(t).$$

- *The conditional variance of the discounted cumulated optimal consumption plus the discounted optimal terminal wealth is given by*[4]

$$Var_{t,x}\left[\int_t^T e^{-\rho(s-t)}c^*(s)ds + e^{-\rho(T-t)}X^{c^*,\pi^*}(T)\right] = \frac{2}{\gamma}(b(t) - B(t)).$$

Here B, b and $b - B$ are given by (33)–(35).

3.2 Discussion of the solution and the problem

As mentioned in Björk et al. (2012) one can argue that from an economic point of view the optimal investment strategy does **not** make sense. From the expression of the optimal investment strategy we see that the optimal amount of money to invest in stocks is independent of wealth, i.e. for a given γ a rich investor and a poor investor optimally invest the same amount of money in stocks. For a one-period model the optimal investment strategy is reasonable since we would expect the richer investor to have a lower value of γ. However, for a multi-period model the strategy seems to be economically unreasonable. If the investor chooses γ such that it reflects his risk aversion corresponding to his initial wealth, then at a later point in time t, due to the progression of wealth, this γ (likely) no longer reflect his risk aversion corresponding to his present wealth $X^{c^*,\pi^*}(t)$. Obviously, the investor should choose his γ in a more sophisticated way. One approach is to let the risk aversion depend on present time and wealth. This case is analyzed in Section 4. The authors think that it is important, even though some might see it as equivalent concepts, to emphasize that we twist the objective function because we realize that the problem, and not the solution, is inappropriate.

We can interpret wealth as a pension saving account and labor income net of consumption as pension contributions. The constraint that the consumption rate c only is allowed to take values between a deterministic upper and lower boundary, $c_{\min}$ respectively $c_{\max}$, has a natural interpretation. If for example we have $c_{\min}(t) = k_1\ell(t)$ and $c_{\max}(t) = k_2\ell(t)$, for constants

[4] We have that

$$Var_{t,x}\left[\int_t^T e^{-\rho(s-t)}c^*(s)ds + e^{-\rho(T-t)}X^{c^*,\pi^*}(T)\right] = F^{(2)}(t,x) - \left(F^{(1)}(t,x)\right)^2 = \frac{2}{\gamma}(b(t) - B(t)).$$

$0 < k_1 < k_2 < 1$, we have that the investor is forced to spend no less than a minimal fraction $1 - k_2$ and no more than a maximal fraction $1 - k_1$ of his labor income on pension contributions. This corresponds to a compulsory pension scheme and a subsistence level, respectively.

From the expression of the optimal consumption strategy we have that the investor optimally consumes the minimum (maximum) allowed if he is patient (impatient). That is, if he has a time preference parameter ρ smaller (greater) than the risk free interest rate r. If we take the optimal investment strategy as given this result is easy to understand: First of all the deterministic consumption strategy minimizes the variance term in (22). If the investor chooses to consume the minimum allowed he saves as much as possible. These savings earn, according to the optimal investment strategy, the risk-free interest rate. This consumption strategy is only optimal if these savings including interest are large enough for the investor to be willing to wait for them, i.e. if $r > \rho$. Reversely, if $r < \rho$ the investor is to impatient to wait for the savings including interest and chooses to consume the maximum allowed instead. However, it is important to emphasize that initial we could not have foreseen this trivial optimal consumption strategy since we search for c^* and π^* simultaneously.

Finally, one should notice that the optimal strategy does not guarantee that wealth stays non-negative (or above any other given lower boundary). This is also the case for the optimal strategy derived in Basak and Chabakauri (2010) and Björk and Murgoci (2009). However, the optimal strategy is perfectly reasonable. In the definition of the problem (22) we do not exclude strategies for which the corresponding wealth has positive probability of becoming negative. Proposition 3.1 simply tells us that, as a consequence hereof, it is optimal to continue to take risky investment decisions and consume even though this may punish the total utility obtained over the interval $[0, T]$ in form of a negative wealth at time T.

4 Mean-variance with time and state dependent risk aversion

By introducing a time and state dependent risk aversion function the mean-variance problem without pre-commitment including consumption and terminal wealth becomes much more complicated. For the case $c = \ell = 0$ this is analyzed, for state (but not time) dependent risk aversion, by Björk et al. (2012). We consider the special case of time and state dependent risk aversion where the investor's risk aversion is hyperbolic in present wealth plus the financial value of future labor income net of consumption. One can argue that the investor, hereby, can influence his own risk aversion by choosing his consumption rate in a certain way. This is in our opinion however perfectly reasonable. All it says is that if the investor knows that he in the future is going to save money by consuming less then he should act as if he already had more money and adapt his risk aversion to that situation. Solving the mean-variance problem with time and state dependent risk aversion we only allow the investor to look for strategies for which the corresponding wealth plus the financial value of future labor income net of consumption stays positive over the entire interval. This conforms with the well-known and often required constraint that wealth plus human capital must stay positive at all times. Consequently, terminal wealth becomes positive. As in Section 3 we restrict consumption by a time dependent upper and lower boundary.

It turns out that the optimal investment strategy becomes linear in the investor's wealth plus financial value of future labor income net of consumption. Furthermore, as in the case of constant risk aversion, the optimal consumption rate becomes a deterministic bang-bang strategy. The deterministic function determining when it is optimal to consume the maximum or minimum allowed is given by a system of non-linear differential equations for which we have no explicit solution. Moreover, the constraint that wealth plus the financial value of future labor income net of consumption must stay positive at all times may become binding. That is, in order to be able to finance his consumption stream, the investor optimally consumes the minimum allowed from the point in time where the constraint (may) becomes active and onwards. Opposed to the case with constant risk aversion we find that it is not always optimal to either consume the maximum or minimum allowed at all times. For some investors we find that it is optimal to first, for a period of time, to consume the maximum allowed and then, for the remainder of the period, to consume the minimum allowed. The results are analyzed in details below.

4.1 Presenting and solving the problem

Define the time-t financial value of future labor income net of consumption by

$$K^{(c)}(t,x) := E_{t,x}^{\widehat{P}}\left[\int_t^T e^{-r(s-t)}\Big(\ell(s) - c\left(s, X^{c,\pi}(s)\right)\Big)ds\right], \tag{36}$$

and define, for a finite time dependent interval $\mathcal{D} \in \mathbb{R}$, the set of strategies

$$\mathcal{B}(t) := \Big\{(c,\pi) \,\Big|\, c(s) \in \mathcal{D}(s) := [c_{\min}(s), c_{\max}(s)],\ X^{c,\pi}(s) + K^{(c)}(s, X^{c,\pi}(s)) \geq 0, \forall s \in [t,T]\Big\}.$$

In (36) the expectation is derived under the martingale measure $\widehat{P}$ defined by (6). The condition $x + K^{(c)} \geq 0$ ensures that the strategies in $\mathcal{B}$ fulfill the natural requirement that, at any time, the investor should be able to finance his own consumption stream. That is, the investor has to be sure that, at any time, consumption can be financed by present wealth, capital gains and labor income. To ensure that the set of strategies $\mathcal{B}$ is non-empty we must assume that, initially, $c_{\min}$ fulfils the condition $x + K^{(c_{\min})} \geq 0$. Note that depending on the size of $c_{\max}$ and the size of labor income, consumption is always either restricted directly by the upper bound $c_{\max}$ *or* indirectly by the more technical constraint $x + K^{(c)} \geq 0$. A lower bound $c_{\min}$ seems natural since the investor is expected to have a subsistence level. At least consumption should naturally be restricted to stay non-negative.

We now consider the problem of finding the optimal strategy for the objective given by

$$\begin{aligned} &E_{0,x_0}\left[\int_0^T e^{-\rho s}c(s)ds + e^{-\rho T}X^{c,\pi}(T)\right] \\ &- \frac{\gamma}{2\left(x_0 + K^{(c)}(0,x_0)\right)} Var_{0,x_0}\left[\int_0^T e^{-\rho s}c(s)ds + e^{-\rho T}X^{c,\pi}(T)\right], \end{aligned} \tag{37}$$

where we restrict the class of admissible strategies to $(c,\pi) \in \mathcal{A}\bigcap\mathcal{B}$. Note that hereby the risk aversion factor on the conditional variance is a function of the consumption stream. The corresponding function f is given by

$$f(t,x,y,z) = y - \frac{\psi(t,x)}{2}\left(z - y^2\right),$$

where $\psi(t,x) = \frac{\gamma}{x + K^{(c)}(t,x)}$. The system of partial differential equations we want to solve in order to obtain the optimal value function and the optimal strategy is given by (17) and (18)[5]. Clearly,

$$\begin{aligned} &f_t = -\frac{\psi_t}{2}\left(z - y^2\right), f_x = -\frac{\psi_x}{2}\left(z - y^2\right), f_{xx} = -\frac{\psi_{xx}}{2}\left(z - y^2\right), \\ &f_y = 1 + \psi y, f_{yy} = \psi, f_z = -\frac{\psi}{2}, f_{xz} = -\frac{\psi_x}{2}, f_{xy} = \psi_x y, \\ &f_{zz} = f_{yz} = 0. \end{aligned}$$

[5]Since $F_t = F_t^{(1)} - \frac{\frac{\partial}{\partial t}\psi(t,x)}{2}\left(F^{(2)} - \left(F^{(1)}\right)^2\right) - \frac{\psi(t,x)}{2}F_t^{(2)} + \psi(t,x)F^{(1)}F_t^{(1)}$ we only need to solve two of the three differential equations given by (16)–(18). We choose to solve (17) and (18).

Combining with the formulas (19)–(21) we obtain, after some rearrangement of terms, the following expressions which turn out to be useful

$$F_x - Q = F_x^{(1)} + \psi F^{(1)} F_x^{(1)} - \frac{\psi}{2} F_x^{(2)}, \tag{38}$$

$$F_{xx} - U = F_{xx}^{(1)} - \frac{\psi}{2} F_{xx}^{(2)} + \psi F^{(1)} F_{xx}^{(1)}, \tag{39}$$

$$J = \rho F^{(1)} - c - \left(\psi\rho + \frac{\psi_t}{2}\right)\left(F^{(2)} - \left(F^{(1)}\right)^2\right). \tag{40}$$

By characterizing (36) as the solution to a (Feynman-Kač) PDE we get that

$$\psi_t = -\frac{\gamma}{\left(x + K^{(c)}\right)^2}\left(rK^{(c)} - \ell + c - (rx + \ell - c)K_x^{(c)} - \frac{1}{2}\pi^2\sigma^2 x^2 K_{xx}^{(c)}\right).$$

Note that the coefficient of $K_x^{(c)}$ does not include the excess return $(\alpha - r)$ since $K^{(c)}$ is defined as a conditional expectation under the martingale measure $\widehat{P}$. Insert this in (40) to obtain

$$\begin{aligned} J = \; & \rho F^{(1)} - c - \frac{\gamma\rho}{x + K^{(c)}}\left(F^{(2)} - \left(F^{(1)}\right)^2\right) + \frac{\gamma}{2\left(x + K^{(c)}\right)^2}\Big(rK^{(c)} - \ell + c \\ & - (rx + \ell - c)K_x^{(c)} - \frac{1}{2}\pi^2\sigma^2 x^2 K_{xx}^{(c)}\Big)\left(F^{(2)} - \left(F^{(1)}\right)^2\right). \end{aligned} \tag{41}$$

We are now ready to derive a candidate for the optimal strategy. To do this we consider the two cases where the constraint $x + K^{(c)} \geq 0$ is non-binding and binding, respectively. That is, we consider, for an arbitrary point (t, x), the two cases $x + K^{(c_{\min})}(t, x) > 0$ and $x + K^{(c_{\min})}(t, x) = 0$, respectively.

The non-binding case ($x + K^{(c_{\min})}(t, x) > 0$)

A candidate for the optimal strategy is found by differentiating with respect to c and π inside the curly brackets in (16). We get

$$c^*(t, x) = \begin{cases} c_{\max}(t), & \text{if } C(t, x) < 1, \\ \text{non-defined}, & \text{if } C(t, x) = 1, \\ c_{\min}(t), & \text{if } C(t, x) > 1, \end{cases} \tag{42}$$

where

$$C(t, x) := F_x(x, t) - Q(x, t) + \frac{\gamma}{2(x + K^{(c^*)}(t))^2}\left(1 + K_x^{(c^*)}(t, x)\right)\left(F^{(2)}(t, x) - \left(F^{(1)}(t, x)\right)^2\right),$$

and

$$\pi^*(t, x)x = -\frac{\alpha - r}{\sigma^2}\frac{F_x - Q}{F_{xx} - U}, \tag{43}$$

(provided $U > F_{xx}$). The optimal consumption strategy is a bang-bang strategy and, for the moment, it appears to be stochastic and dependent on wealth. However, we now search for a solution to the problem in the set of solutions such that the optimal consumption strategy becomes deterministic. If we find such a solution this is of course no restriction. Thus, we search for solutions where $F^{(1)}$ and $F^{(2)}$ are in a form designed exactly such that this is the case. We propose that

$$\begin{aligned} F^{(1)}(t, x) &= a(t)\left(x + K^{(c^*)}(t)\right) + b(t), \\ F^{(2)}(t, x) &= f(t)\left(x + K^{(c^*)}(t)\right)^2 + g(t)\left(x + K^{(c^*)}(t)\right) + h(t), \end{aligned}$$

where a, b, f, g and h are deterministic functions of time, where the candidate for the optimal consumption strategy, c, is assumed to be independent of wealth, and where[6]

$$a(t)b(t) = \frac{g(t)}{2}, \tag{44}$$

$$h(t) = b(t)^2. \tag{45}$$

In this case we get that

$$K^{(c^*)}(t) = \int_t^T e^{-r(s-t)}(\ell(s) - c^*(s))ds.$$

Due to (44) and (45) we get that the variance term in the value function in (37) has the form

$$\begin{aligned}
& F^{(2)} - \left(F^{(1)}\right)^2 \\
= \; & \left(f(t) - a(t)^2\right)\left(x + K^{(c^*)}(t)\right)^2 + 2\left(\frac{g(t)}{2} - a(t)b(t)\right)\left(x + K^{(c^*)}(t)\right) + h(t) - b(t)^2 \\
= \; & \left(f(t) - a(t)^2\right)\left(x + K^{(c^*)}(t)\right)^2.
\end{aligned} \tag{46}$$

The form of F is completely determined by $F^{(1)}$ and $F^{(2)}$. Using (46) we get that

$$\begin{aligned}
F(t,x) &= F^{(1)} - \frac{\gamma}{2\left(x + K^{(c^*)}(t)\right)}\left\{F^{(2)} - \left(F^{(1)}\right)^2\right\} \\
&= a(t)\left(x + K^{(c^*)}(t)\right) + b(t) - \frac{\gamma}{2}\left(f(t) - a(t)^2\right)\left(x + K^{(c^*)}(t)\right).
\end{aligned}$$

The partial derivatives of interest become

$$\begin{aligned}
F_t^{(1)} &= a'(t)\left(x + K^{(c^*)}(t)\right) + a(t)\left(rK^{(c^*)}(t) - \ell(t) + c(t)\right) + b'(t), \\
F_x^{(1)} &= a(t), \; F_{xx}^{(1)} = 0, \\
F_t^{(2)} &= f'(t)\left(x + K^{(c^*)}(t)\right)^2 + 2f(t)\left(rK^{(c^*)}(t) - \ell(t) + c(t)\right)\left(x + K^{(c^*)}(t)\right) \\
&\quad + g'(t)\left(x + K^{(c^*)}(t)\right) + g(t)\left(rK^{(c^*)}(t) - \ell(t) + c(t)\right) + h'(t), \\
F_x^{(2)} &= 2f(t)\left(x + K^{(c^*)}(t)\right) + g(t), \; F_{xx}^{(2)} = 2f(t).
\end{aligned}$$

Inserting this in (38) gives

$$\begin{aligned}
F_x(t,x) - Q(t,x) &= a(t) + \frac{\gamma}{x + K^{(c^*)}(t)}\left[a(t)^2\left(x + K^{(c^*)}(t)\right) + a(t)b(t)\right] \\
&\quad - \frac{\gamma}{2\left(x + K^{(c^*)}(t)\right)}\left[2f(t)\left(x + K^{(c^*)}(t)\right) + g(t)\right] \\
&= a(t) + \gamma\left(a(t)^2 - f(t)\right) + \frac{\gamma}{x + K^{(c^*)}(t)}\left(a(t)b(t) - \frac{g(t)}{2}\right).
\end{aligned}$$

By assumption (44) this reduces to

$$F_x(t,x) - Q(t,x) = a(t) + \gamma\left(a(t)^2 - f(t)\right). \tag{47}$$

[6]The assumptions given in (44) and (45) turn out to be consistent with the assumption that the candidate for the optimal consumption strategy is deterministic.

Inserting the partial derivatives in (39) gives

$$F_{xx}(t,x) - U(t,x) = -\frac{\gamma f(t)}{x + K^{(c^*)}(t)}. \tag{48}$$

Now, insert $K_x^{(c^*)} = 0$, (46) and (47) in (42) to obtain

$$c^*(t) = \begin{cases} c_{\max}(t), & \text{if } a(t) + \frac{\gamma}{2}\left(a(t)^2 - f(t)\right) < 1, \\ \text{non-defined}, & \text{if } a(t) + \frac{\gamma}{2}\left(a(t)^2 - f(t)\right) = 1, \\ c_{\min}(t), & \text{if } a(t) + \frac{\gamma}{2}\left(a(t)^2 - f(t)\right) > 1, \end{cases} \tag{49}$$

and insert (47) and (48) in (43) to obtain

$$\pi^*(t,x)x = \frac{\alpha - r}{\sigma^2 \gamma f(t)} \left[a(t) + \gamma\left(a(t)^2 - f(t)\right)\right]\left(x + K^{(c^*)}(t)\right), \tag{50}$$

provided that[7]

$$\frac{\gamma f(t)}{x + K^{(c^*)}(t)} > 0. \tag{51}$$

Now plug in (49), (50) and the relevant derivatives into (17) and include the terminal conditions to obtain

$$\boxed{\begin{aligned} a_t &= -\left\{(r-\rho) + \frac{(\alpha - r)^2}{\sigma^2 \gamma f}\left[a + \gamma(a^2 - f)\right]\right\} a, \\ a(T) &= 1, \\ b_t &= -c^* + \rho b, \\ b(T) &= 0. \end{aligned}} \tag{52}$$

Now, insert (49), (50) and the relevant derivatives into (18) and include the terminal conditions to obtain

$$\boxed{\begin{aligned} f_t &= -\left\{2\left((r-\rho) + \frac{(\alpha - r)^2}{\sigma^2 \gamma f}\left[a + \gamma(a^2 - f)\right]\right) + \frac{(\alpha - r)^2}{\sigma^2 \gamma^2 f^2}\left[a + \gamma(a^2 - f)\right]^2\right\} f, \\ g_t &= -\left(r + \frac{(\alpha - r)^2}{\sigma^2 \gamma f}\left[a + \gamma(a^2 - f)\right]\right) g - 2c^* a + 2\rho g, \\ h_t &= -2c^* b + 2\rho h, \\ f(T) &= 1, \\ g(T) &= 0, \\ h(T) &= 0. \end{aligned}} \tag{53}$$

From (52) and (53) we immediately obtain the solutions

$$\begin{aligned} b(t) &= \int_t^T e^{-\rho(s-t)} c^*(s) ds, \\ h(t) &= \left(\int_t^T e^{-\rho(s-t)} c^*(s) ds\right)^2, \end{aligned}$$

i.e. assumption (45) is indeed fulfilled. Finally, we have three things left to verify!

[7]The candidate of the optimal investment strategy (43) was derived under the condition that $U > F_{xx}$. By use of (48) we can write this condition as (51).

- We need to show that assumption (44) is fulfilled.
- The candidate of the optimal investment strategy (50) was derived under the condition (51). This assumption has to be verified.
- The non-linear system of partial differential equations given by (52) and (53) does not satisfy the usual Lipschitz and growth conditions. Global existence and uniqueness are therefore not guaranteed. We need to show that the system of partial differential equations in fact has a unique solution.

This is all done in Appendix A.2.

The binding case $(x + K^{(c_{\min})}(t,x) = 0)$

Whenever the constraint is active the investor is forced to consume at the minimum rate allowed. By the definition of $K^{(c)}$ given by (36) the only investment strategy which can finance this consumption stream, while keeping $x + K^{(c)} \geq 0$, is $\pi = 0$. We get that the only strategy, and thereby the optimal strategy, in $\mathcal{A} \bigcap \mathcal{B}$ is $c^*(t) = c_{\min}(t)$ and $\pi^*(t,x) = 0$. Obviously, once the restriction becomes active it becomes binding for the remaining time of the period. That is, if $x + K^{(c)}(t,x) = 0$ we get $X^{c,\pi}(s) + K^{(c)}(s, X^{c,\pi}(s)) = 0$ for all $s \in [t,T]$.

Collecting the results from the two cases we summarize as follows

Proposition 4.1. *For the mean-variance problem given by (37) we have the following results.*

- *The optimal strategy is given by*

$$c^*(t) = \begin{cases} \widehat{c}(t), & \text{if } t \in [0, t^*), \\ c_{\min}(t), & \text{if } t \in [t^*, T], \end{cases}$$
$$\pi^*(t,x)x = \frac{\alpha - r}{\sigma^2 \gamma f(t)} \left[a(t) + \gamma \left(a(t)^2 - f(t) \right) \right] \left(x + K^{(c^*)}(t) \right),$$

where

$$\widehat{c}(t) = \begin{cases} c_{\max}(t), & \text{if } a(t) + \frac{\gamma}{2} \left(a(t)^2 - f(t) \right) < 1, \\ \text{non-defined}, & \text{if } a(t) + \frac{\gamma}{2} \left(a(t)^2 - f(t) \right) = 1, \\ c_{\min}(t), & \text{if } a(t) + \frac{\gamma}{2} \left(a(t)^2 - f(t) \right) > 1, \end{cases}$$

and t^* is given by

$$\sup \left\{ t^* \in [t, T] \,\middle|\, X^{c,\pi}(t) + \int_t^{t^*} e^{-r(s-t)} (\ell(s) - \widehat{c}(s)) ds + \int_{t^*}^T e^{-r(s-t)} (\ell(s) - c_{\min}(s)) ds \geq 0 \right\}. \tag{54}$$

- *The optimal value function is given by*

$$V(t,x) = a(t) \left(x + K^{(c^*)}(t) \right) + b(t) - \frac{\gamma}{2} \left(f(t) - a(t)^2 \right) \left(x + K^{(c^*)}(t) \right).$$

- *The conditional expected value of the discounted cumulated optimal consumption plus the discounted optimal terminal wealth is given by*

$$E_{t,x} \left[\int_t^T e^{-\rho(s-t)} c^*(s) ds + e^{-\rho(T-t)} X^{c^*,\pi^*}(T) \right] = a(t) \left(x + K^{(c^*)}(t) \right) + b(t).$$

- *The conditional variance of the discounted cumulated optimal consumption plus the discounted optimal terminal wealth is given by*

$$Var_{t,x} \left[\int_t^T e^{-\rho(s-t)} c^*(s) ds + e^{-\rho(T-t)} X^{c^*,\pi^*}(T) \right] = \left(f(t) - a(t)^2 \right) \left(x + K^{(c^*)}(t) \right)^2.$$

Here,

$$K^{(c^*)}(t) = \int_t^T e^{-r(s-t)} (\ell(s) - c^*(s)) ds,$$

and a, b, f, g and h are given by the non-linear system of partial differential equations given by (52) and (53).

Remark 4.1. We get that (54) can be written as

$$\sup\left\{t^*\in[0,T]\;\middle|\;x_0+\int_0^{t^*}e^{-rs}(\ell(s)-\widehat{c}(s))ds+\int_{t^*}^T e^{-rs}(\ell(s)-c_{\min}(s))ds\ge 0\right\}.$$

That is, t^* can be found initially at time $t=0$ and will not change at a later point in time. To make this clear consider, at time $t=0$, the two situations:

1. The cases $t^*<T$ and $t^*=T$ with $x_0+K^{(c^*)}(0,x_0)=0$: As already noted, in those cases, the constraint $x+K^{(c)}\ge 0$ becomes binding in the sense that the only investment strategy which can finance the consumption stream c^*, while keeping $x+K^{(c)}\ge 0$, is $\pi^*(t)=0$, $\forall t\in[0,T]$. Obviously, we get $X^{c^*,\pi^*}(t)+K^{(c^*)}(t,X^{c^*,\pi^*}(t))=0$, $\forall t\in[0,T]$, i.e. t^* does not change at a later point in time.
2. The case $t^*=T$ with $x_0+K^{(c^*)}(0,x_0)>0$: We get due to π^* being linear in $x+K^{(c^*)}$ that $X^{c^*,\pi^*}(t)+K^{(c^*)}(t,X^{c^*,\pi^*}(t))>0$, $\forall t\in[0,T]$[8], i.e. $t^*=T$ at all future time points.

We emphasize that it is the binding nature of the constraint together with the dynamic investment strategy that makes t^*, and consequently the optimal consumption strategy, deterministic. Once again, the optimal consumption strategy $c^*(t)$, $t\in[0,T]$, is completely known at time $t=0$.

Remark 4.2. For big enough values of γ we may have (if $c_{\max}$ is big relative to x_0) that $x_0+K^{(c^*)}(0,x_0)=0$ (see Figure 2). One may argue that the optimal value function is then not well-defined. The concern is about the risk aversion function ψ being non-defined (division by zero). Naturally, the strategy (c^*,π^*) defined by Proposition 4.1 is given as the limit of the series of strategies $(c^*_n,\pi^*)_{n=1,2,\ldots}$ where the expression in (54) is strictly positive but tends to zero. From Appendix A.2 formula (90) we have that $x+K^{(c^*_n)}$ follows a geometric Brownian motion. More precise, $X^{c^*_n,\pi^*}(T)=(x+K^{(c^*_n)}(t,x))\exp(\ldots)$ where the stochastic exponential term depends on the dynamics of W. We conclude that the value function is well-defined since

$$\begin{aligned}
&E_{0,x_0}\left[\int_0^T e^{-\rho s}c^*_n(s)ds+e^{-\rho T}X^{c^*_n,\pi^*}(T)\right]\\
&-\frac{\gamma}{2\left(x_0+K^{(c^*_n)}(0,x_0)\right)}Var_{0,x_0}\left[\int_0^T e^{-\rho s}c^*_n(s)ds+e^{-\rho T}X^{c^*_n,\pi^*}(T)\right]\\
=&\int_0^T e^{-\rho s}c^*_n(s)ds+\underbrace{\left(x_0+K^{(c^*_n)}(0,x_0)\right)}_{\searrow 0}\exp(\ldots)-\underbrace{\frac{\gamma\left(x_0+K^{(c^*_n)}(0,x_0)\right)^2}{2\left(x_0+K^{(c^*_n)}(0,x_0)\right)}}_{\searrow 0}\underbrace{Var_{0,x_0}\left[\exp(\ldots)\right]}_{\searrow 0}\\
\longrightarrow&\int_0^T e^{-\rho s}c^*(s)ds,
\end{aligned}$$

which coincides with Proposition 4.1.

[8]See Appendix A.2 formula (90) where we, for the non-binding case, show that $x+K^{(c^*)}$ is a geometric Brownian motion.

4.2 Discussion of the solution

The investment strategy

From the expression of the optimal investment strategy we see that the optimal amount of money to invest in stocks is proportional to wealth plus the financial value of future labor income net of consumption. From an economic point of view this seems to be a fairly reasonable investment strategy. First of all, a rich investor should invest more in stocks than a poor investor. Second, if we know that a large amount of money will be injected continuously into the savings, then we should also invest a large amount of money in stocks.

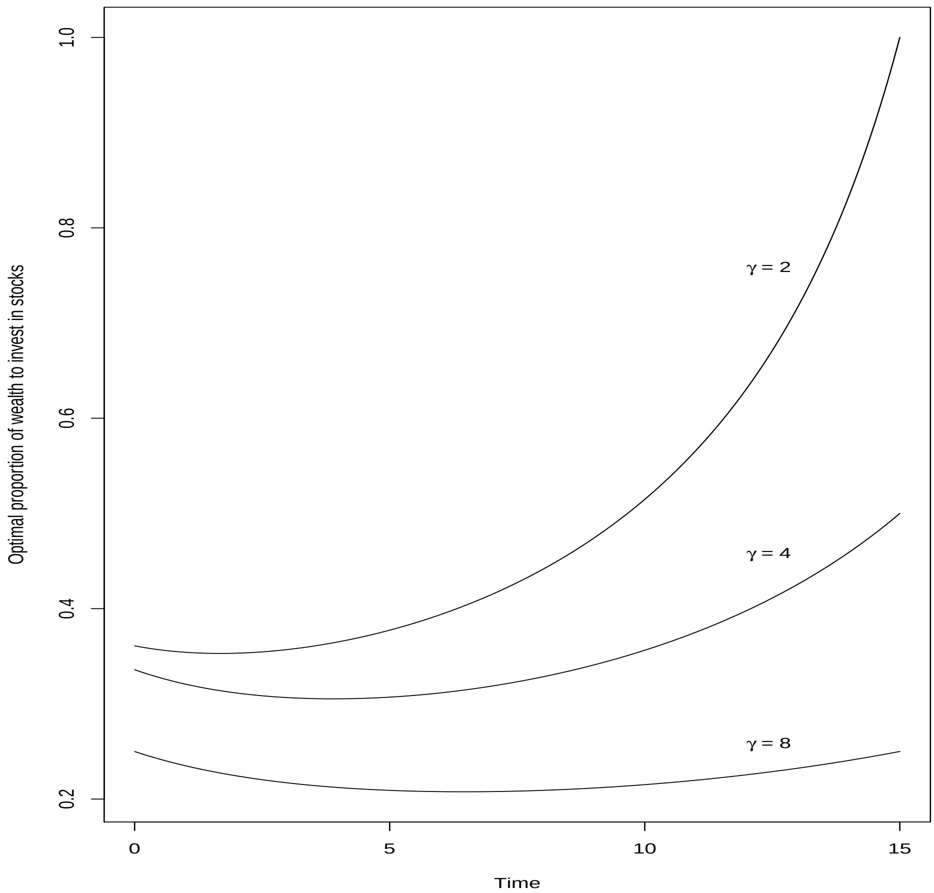

Figure 1: Parameter values are $T = 15$, $r = 0.04$, $\alpha = 0.12$, $\sigma = 0.2$ and $\rho = 0.02$. The initial wealth is $X_0 = 1000000$ DKK, labor income is during the hole time period 30000 DKK/month and the minimal consumption allowed (which turns out to equal the optimal consumption for all three choices of γ) is during the hole time period 21000 DKK/month.

Two important questions are:

- How does the optimal proportion of wealth to invest in stocks develop as time goes by?
- How do shifts in parameter values (γ in particular) influence the optimal proportion of wealth to invest in stocks?

In order to answer those questions define

$$\widetilde{\pi}^*(t) = \frac{\alpha - r}{\sigma^2 \gamma f(t)} \left[a(t) + \gamma \left(a(t)^2 - f(t) \right) \right].$$

Then

$$\pi^*(t,x)x = \widetilde{\pi}^*(t) \left(x + K^{(c^*)}(t) \right).$$

In Appendix A.2 we show that we have the following integral equation for $\widetilde{\pi}^*$:

$$\widetilde{\pi}^*(t) = \frac{\alpha - r}{\sigma^2 \gamma} \left\{ e^{-\int_t^T \left[(r-\rho) + (\alpha - r)\widetilde{\pi}^*(s) + \sigma^2 \widetilde{\pi}^*(s)^2 \right] ds} + \gamma e^{-\int_t^T \sigma^2 \widetilde{\pi}^*(s)^2 ds} - \gamma \right\}.$$

We recognize this integral equation from Björk et al. (2012) who derive this for the case without consumption and labor income. If $r > \rho$ we conclude that $\widetilde{\pi}^*$ is increasing in time. On the other hand we have, for the non-binding case, that $(X^{c^*,\pi^*}(t) + K^{(c^*)}(t))/X^{c^*,\pi^*}(t)$ is expected to decrease with time. How fast $\widetilde{\pi}^*$ increases and how fast $(X^{c^*,\pi^*}(t) + K^{(c^*)}(t))/X^{c^*,\pi^*}(t)$ decreases depends in a complex way on the value of γ[9]. From the optimization problem (37) we can argue that (since a smaller value of γ corresponds to giving the variance term a smaller weight) we expect a smaller value of γ to imply a more aggressive investment strategy in general, i.e. a larger value of $\widetilde{\pi}$. Due to the complexity of the system of partial differential equations given in (52) and (53) it seems difficult to prove this. For a average scenario it is in Figure 1 illustrated that a smaller γ indeed implies a more aggressive investment strategy. For small values of γ we also note that we should indeed expect the optimal proportion of wealth to invest in stocks to increase over time. However, as seen in Figure 1 ($\gamma = 8$), we also have that for γ large enough the optimal proportion of wealth to invest in stocks seems to be approximately constant.

The consumption strategy

As already mentioned the upper and lower boundary for the consumption rate has a natural interpretation when we think of wealth as a pension saving account (see Subsection 3.2).

Let us consider the non-binding case ($x_0 + K^{(c^*)}(0, x_0) > 0$). From the expression of the optimal consumption strategy we see that it is optimal either to consume the maximum or minimum allowed dependent on whenever the deterministic expression $a(t) + \frac{\gamma}{2}\left(a(t)^2 - f(t)\right)$ is smaller or larger than 1. Since the functions a and f are given by the non-linear system of differential equations (52) and (53) it is difficult to analyze the optimal consumption strategy.

[9] $K^{(c^*)}$ depends on γ since obviously the optimal consumption strategy does so.

However, some insight can be obtained by the following calculations:

$$
\begin{aligned}
& \frac{\partial}{\partial t}\Big(a(t)+\frac{\gamma}{2}(a(t)^2-f(t))\Big) \\
= \;& a'(t)+\gamma a(t)a'(t)-\frac{\gamma}{2}f'(t) \\
= \;& -\Big\{(r-\rho)+\frac{(\alpha-r)^2}{\sigma^2\gamma f(t)}\big[a(t)+\gamma(a(t)^2-f(t))\big]\Big\}a(t) \\
& -\Big\{(r-\rho)+\frac{(\alpha-r)^2}{\sigma^2\gamma f(t)}\big[a(t)+\gamma(a(t)^2-f(t))\big]\Big\}\gamma a(t)^2 \\
& +\Big\{(r-\rho)+\frac{(\alpha-r)^2}{\sigma^2\gamma f(t)}\big[a(t)+\gamma(a(t)^2-f(t))\big]\Big\}\gamma f(t) \\
& +\frac{(\alpha-r)^2}{\sigma^2\gamma^2 f(t)^2}\big[a(t)+\gamma(a(t)^2-f(t))\big]^2\frac{\gamma}{2}f(t) \\
= \;& -\Big\{(r-\rho)+\frac{(\alpha-r)^2}{\sigma^2\gamma f(t)}\big[a(t)+\gamma(a(t)^2-f(t))\big]\Big\}\big[a(t)+\gamma\big(a(t)^2-f(t)\big)\big] \\
& +\frac{1}{2}\frac{(\alpha-r)^2}{\sigma^2\gamma f(t)}\big[a(t)+\gamma(a(t)^2-f(t))\big]^2 \\
= \;& -(r-\rho)\big[a(t)+\gamma\big(a(t)^2-f(t)\big)\big]-\frac{1}{2}\frac{(\alpha-r)^2}{\sigma^2\gamma f(t)}\big[a(t)+\gamma(a(t)^2-f(t))\big]^2. \qquad (55)
\end{aligned}
$$

In order to allow for an intuitive interpretation of the consumption strategy we *assume* in the following that $a(t)+\gamma\big(a(t)^2-f(t)\big)>0$. Note that this corresponds to assuming that the optimal amount of money to invest in stocks is strictly positive at all time points[10]. By (95) we have that $f(t)>0$ and we can for the case $r\geq\rho$ conclude that $\frac{\partial}{\partial t}\big(a(t)+\frac{\gamma}{2}(a(t)^2-f(t))\big)<0,\ \forall t\in[0,T]$. Since we also have the terminal condition $\big(a(T)+\frac{\gamma}{2}(a(T)^2-f(T))\big)=1$ we can now make the following statement:

$$
r\geq\rho\Longrightarrow\Big(a(t)+\frac{\gamma}{2}(a(t)^2-f(t))\Big)>1\Longrightarrow c^*(t)=c_{\min}(t),\ \forall t\in[0,T). \qquad (56)
$$

If $r<\rho$ the expression (55) consists of a positive term minus a positive term, and we can therefore not make any conclusions about the optimal consumption behavior. Figure 2 shows how the optimal consumption strategy looks in the case of $r<\rho$. We see that for γ small enough it is optimal to consume the minimum allowed during the whole time period, and for γ large enough it is optimal to consume the maximum allowed during the whole time period. For certain values of γ we see that it is optimal in the beginning to consume the maximum allowed and then later to consume the minimal allowed.

In order to get a better understanding of the optimal consumption strategy we make the following observation. If we take the optimal investment strategy as given we can try to comment on the result given by (56) and Figure 2. First of all, in contrast to the case with constant risk aversion, the deterministic consumption strategy does influence the variance term. If we, during an infinitesimal time interval $[t,t+dt]$, choose to consume the minimum allowed we save the amount of money $(c_{\max}-c_{\min})dt$ in addition to $(\ell-c_{\max})\,dt$. The fraction $\widetilde{\pi}^*$ of these money is invested in stocks (which do contribute to the variance term) and the rest is invested in the bank account (which do not contribute to the variance term).

For the case $r\geq\rho$ the expected investment return on the saved amount of money is larger than ρ, and the mean term in (37) can therefore, in terms of consumption, be maximized by

[10]The optimal investment strategy is given by (50). From (90) we have (for the non-binding case) that $X^{c^*,\pi^*}(t)+K^{(c^*)(t)}>0,\ t\in[0,T]$, and it follows that the optimal amount of money to invest in stocks is strictly positive over the interval $[0,T]$ iff $a(t)+\gamma\big(a(t)^2-f(t)\big)>0,\ \forall t\in[0,T]$. Evidence from discrete time calculations leads us to conjecture that the former, and thereby the latter, is true. However, this is a topic outside the scope of the paper which calls for further research.

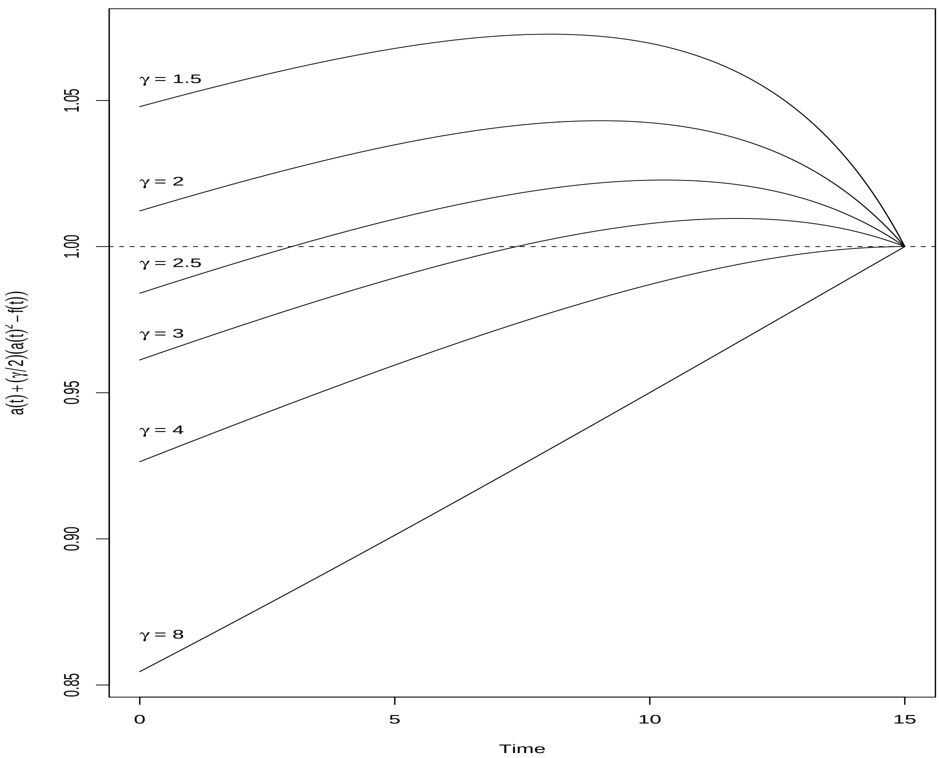

Figure 2: Parameter values are $T = 15,\ r = 0.04,\ \alpha = 0.12,\ \sigma = 0.2$ and $\rho = 0.06$. Points falling below $y = 1$ corresponds to an optimal consumption rate equal to the maximum allowed and points falling above $y = 1$ corresponds to an optimal consumption rate equal to the minimum allowed.

choosing the minimum consumption rate allowed. We also have that a minimum consumption rate minimizes the variance weight term $\gamma/2(x + K^{(c)}(t))$, since a minimum consumption rate maximizes $K^{(c)}$. On the other hand, choosing to consume the minimum allowed also maximizes the variance term in (37) through the investment. From (56) we can conclude that maximizing the mean term and minimizing the variance weight term (in terms of consumption) makes it more than up for a larger variance term (in terms of consumption through the investment strategy).

The chain of reasoning seems to stay true for the case $r < \rho$. In this case the expected return on the saved money is larger than ρ if and only if $\widetilde{\pi}^*$ is big enough. From Figure 1 we have that this is the case for a small enough value of γ. Correspondingly, Figure 2 shows that it is optimal to consume the minimum (maximum) allowed for a small (large) enough value of γ. The reason that it is optimal for an investor with a given value of γ (not too large neither too small) first to consume the maximum allowed and after some time the minimum allowed is due to the fact that $\widetilde{\pi}^*$ is increasing in time.

5 Mean-standard deviation without pre-commitment

In this section we study and solve the mean-standard deviation problem without pre-commitment including consumption and terminal wealth. We consider a general risk aversion function ψ, which we assume fulfills some reasonable constraints, and show that the optimal investment strategy becomes the same ($\pi^* = 0$) for all variants of ψ. Finally, we give an interpretation of the optimal strategy which also helps us understand the fundamental difference between pre-commitment and without pre-commitment.

5.1 Presenting and solving the problem

Consider the problem of finding the optimal strategy for the objective given by

$$E_{0,x_0}\left[\int_0^T e^{-\rho s}c(s)ds + e^{-\rho T}X^{c,\pi}(T)\right] - \psi(0,x_0)\left(Var_{0,x_0}\left[\int_0^T e^{-\rho s}c(s)ds + e^{-\rho T}X^{c,\pi}(T)\right]\right)^{\frac{1}{2}}, \tag{57}$$

where $\psi \in \mathcal{C}^{1,2}$, $\psi > 0$, is a risk aversion function with ψ_x, ψ_{xx} and ψ_t finite $\forall (t,x) \in [0,T] \times \mathbb{R}$, and where we again restrict the admissible strategies to $(c,\pi) \in \mathcal{A} \bigcap (\mathcal{D} \times \mathbb{R})$, where $\mathcal{D}(s) = [c_{\min}(s), c_{\max}(s)]$, $s \in [0,T]$, is a finite time dependent interval. The corresponding function f is given by

$$f(t,x,y,z) = y - \psi(t,x)\left(z - y^2\right)^{\frac{1}{2}}.$$

The system of partial differential equations we want to solve in order to obtain the optimal value function and the optimal strategy is given by (16) and (17)[11]. Clearly,

$$\begin{aligned}
f_t &= -\psi_t\left(z-y^2\right)^{\frac{1}{2}},\ f_x = -\psi_x\left(z-y^2\right)^{\frac{1}{2}},\ f_{xx} = -\psi_{xx}\left(z-y^2\right)^{\frac{1}{2}},\\
f_y &= 1 + \psi\left(z-y^2\right)^{\frac{-1}{2}}y,\ f_{yy} = \psi\left(z-y^2\right)^{\frac{-3}{2}}y^2 + \psi\left(z-y^2\right)^{\frac{-1}{2}},\\
f_z &= -\frac{\psi}{2}\left(z-y^2\right)^{\frac{-1}{2}},\ f_{zz} = \frac{\psi}{4}\left(z-y^2\right)^{\frac{-3}{2}},\\
f_{xz} &= -\frac{\psi_x}{2}\left(z-y^2\right)^{\frac{-1}{2}},\ f_{xy} = \psi_x\left(z-y^2\right)^{\frac{-1}{2}}y,\ f_{yz} = \frac{-\psi}{2}\left(z-y^2\right)^{\frac{-3}{2}}y.
\end{aligned}$$

Inserting this in (19)–(21) we obtain the following expression

$$Q = -\psi_x\left(F^{(2)} - \left(F^{(1)}\right)^2\right)^{\frac{1}{2}}, \tag{58}$$

$$\begin{aligned}
U = {} & \frac{\psi}{4}\left(F^{(2)} - \left(F^{(1)}\right)^2\right)^{-\frac{3}{2}}\left(F_x^{(2)} - 2F^{(1)}F_x^{(1)}\right)^2 + \psi\left(F^{(2)} - \left(F^{(1)}\right)^2\right)^{-\frac{1}{2}}\left(F_x^{(1)}\right)^2\\
& - \psi_{xx}\left(F^{(2)} - \left(F^{(1)}\right)^2\right)^{\frac{1}{2}} - \psi_x\left(F^{(2)} - \left(F^{(1)}\right)^2\right)^{-\frac{1}{2}}\left(F_x^{(2)} - 2F^{(1)}F_x^{(1)}\right),
\end{aligned} \tag{59}$$

$$J = \rho F^{(1)} - c - \psi\rho\left(F^{(2)} - \left(F^{(1)}\right)^2\right)^{\frac{1}{2}} - \psi_t\left(F^{(2)} - \left(F^{(1)}\right)^2\right)^{\frac{1}{2}}. \tag{60}$$

We are now ready to derive a candidate for the optimal strategy. Again this is done by differentiating with respect to c and π inside the curly brackets in (16). We get

$$c^*(t,x) = \begin{cases} c_{\max}(t), & \text{if } C(t,x) < 0,\\ \text{non-defined}, & \text{if } C(t,x) = 0,\\ c_{\min}(t), & \text{if } C(t,x) > 0, \end{cases} \tag{61}$$

[11] Since $F_t = F_t^{(1)} - \frac{\partial}{\partial t}\psi(t,x)\left(F^{(2)} - \left(F^{(1)}\right)^2\right)^{\frac{1}{2}} - \frac{\psi(t,x)}{2}\left(F^{(2)} - \left(F^{(1)}\right)^2\right)^{-\frac{1}{2}}\left(F_t^{(2)} - 2F^{(1)}F_t^{(1)}\right)$ we only need to solve two of the three differential equations given by (16)–(18). We choose to solve (16) and (17).

where

$$C(t,x) := F_x(x,t) - Q(x,t) - 1 - \left(\frac{\partial}{\partial c}\psi_t\right)\left(F^{(2)} - \left(F^{(1)}\right)^2\right)^{\frac{1}{2}},$$

and

$$\pi^*(t,x)x = -\frac{\alpha - r}{\sigma^2}\frac{F_x - Q}{F_{xx} - U}, \tag{62}$$

(provided $U > F_{xx}$). We now search for solutions in the form,

$$F(t,x) = A(t)x + B(t),$$
$$F^{(1)}(t,x) = a(t)x + b(t),$$

where A, B, a, and b are deterministic functions of time. The derivatives of interest are

$$F_t = A'(t)x + B'(t),\ F_x = A(t),\ F_{xx} = 0,$$
$$F_t^{(1)} = a'(t)x + b'(t),\ F_x^{(1)} = a(t),\ F_{xx}^{(1)} = 0.$$

Inserting this into the system of partial differential equations given by (16) and (17) we find

$$A_t x + B_t = -rx(A-Q) - \frac{1}{2}\pi^*(\alpha - r)x(A-Q) - \ell(A-Q) + c^*(A-Q) + J,$$
$$a_t x + b_t = -rxa - \pi^*(\alpha - r)xa - \ell a + c^*(a-1) + \rho(ax+b).$$

Quite surprisingly, this system of differential equations is solved by $\pi^* = 0$ via the parametrization $F^{(2)} = \left(F^{(1)}\right)^2$ [12]. Note that for this solution, actually U is infinite. However since $\pi^* U$ is finite, the solution is admissible. With $\pi^* = 0$ the system of partial differential equations reduces to

$$\begin{aligned} A_t x + B_t &= -rxA + \rho xa - \ell A + c^*(A-1) + \rho b,\\ A(T) &= 1,\\ B(T) &= 0,\\ a_t x + b_t &= -(r-\rho)xa - \ell a + c^*(a-1) + \rho b,\\ a(T) &= 1,\\ b(T) &= 0, \end{aligned}$$

where we have added the terminal conditions. We obtain the solutions

$$A(t) = a(t) = e^{(r-\rho)(T-t)},$$

[12] $F^{(2)} = \left(F^{(1)}\right)^2$ implies that $F_x^{(2)} - 2F^{(1)}F_x^{(1)} = 0$. Inserting this into (58) and (59) gives

$$Q = 0,$$
$$U = \psi(t,x)\left(F^{(2)} - \left(F^{(1)}\right)^2\right)^{-\frac{1}{2}}\left(F_x^{(1)}\right)^2.$$

Formula (62) now reduces to

$$\pi^*(t,x)x = -\frac{\alpha - r}{\sigma^2}\frac{A(t)\left(F^{(2)} - \left(F^{(1)}\right)^2\right)^{\frac{1}{2}}}{\psi(t,x)\left(F_x^{(1)}\right)^2}.$$

It is then clear that

$$\left\{\left(F^{(2)} = \left(F^{(1)}\right)^2\right) \Rightarrow (\pi^* = 0)\right\} \Leftrightarrow \left\{F_x^{(1)} \neq 0 \text{ and } A(t) \text{ bounded from above}\right\}.$$

Below we show that $F_x^{(1)} = a(t) = A(t) = e^{r(T-t)} > 0$, and clearly we get $\pi^* = 0$.

and

$$B(t) = b(t) = e^{\rho t} \int_t^T \left[\ell(s) e^{(r-\rho)(T-s)} - c^*(s) \left(e^{(r-\rho)(T-s)} - 1 \right) \right] e^{-\rho s} ds.$$

The optimal strategy now follows directly by plugging in the solutions for A and a together with the partial derivatives and the relation $F^{(2)} = \left(F^{(1)}\right)^2$ into (61) and (62). We summarize the results as follows.

Proposition 5.1. *For the mean-standard deviation problem given by (57) we have the following results.*

- *The optimal strategy is given by*

$$c^*(t,x) = \begin{cases} c_{\max}(t), & \text{if } r < \rho, \\ \text{non-defined}, & \text{if } r = \rho, \\ c_{\min}(t), & \text{if } r > \rho, \end{cases}$$
$$\pi^*(t,x) = 0.$$

- *The optimal value function is given by*

$$V(t,x) = e^{(r-\rho)(T-t)} x + B(t).$$

- *The conditional expected value of the discounted cumulated optimal consumption plus the discounted optimal terminal wealth is given by*

$$E_{t,x}\left[\int_t^T e^{-\rho(s-t)} c^*(s) ds + e^{-\rho(T-t)} X^{c^*,\pi^*}(T) \right] = e^{(r-\rho)(T-t)} x + b(t).$$

- *The conditional variance of the discounted cumulated optimal consumption plus the discounted optimal terminal wealth is given by*

$$Var_{t,x}\left[\int_t^T e^{-\rho(s-t)} c^*(s) ds + e^{-\rho(T-t)} X^{c^*,\pi^*}(T) \right] = 0.$$

Here

$$B(t) = b(t) = e^{\rho t} \int_t^T \left[\ell(s) e^{(r-\rho)(T-s)} - c^*(s) \left(e^{(r-\rho)(T-s)} - 1 \right) \right] e^{-\rho s} ds.$$

5.2 Discussion of the solution

The optimal consumption strategy coincides with the one found in Chapter 3. We refer to Subsection 3.2 for an interpretation.

The following intuitive explanation of the somehow surprising optimal investment strategy ($\pi^* = 0$) actually provides us with some helpful insight to understand the nature of inconsistent stochastic optimization problems. In *discrete* time, the definition of a subgame Nash equilibrium strategy is: Consider n players and split the time interval $[T_0, T_n]$ in n equally long intervals. Let player m, $1 \leq m \leq n$, decide on the strategy (c^*_{m-1}, π^*_{m-1}) used over the interval $[T_{m-1}, T_m)$.

- The equilibrium control (c^*_{n-1}, π^*_{n-1}) is obtained by letting player n optimize the value function at time T_{n-1}.
- The equilibrium control (c^*_{n-2}, π^*_{n-2}) is obtained by letting player $n-1$ optimize the value function at time T_{n-2} given the knowledge that player number n will use the strategy (c^*_{n-1}, π^*_{n-1}).

- Proceed recursively by induction.

Now, consider a single period pure portfolio optimization problem with time horizon Δt given by

$$V(0,x) = \sup_{\pi} \left\{ E_{0,x}\left[X^{\pi}(\Delta t)\right] - \psi(0,x)\left(Var_{0,x}\left[X^{\pi}(\Delta t)\right]\right)^{\frac{1}{2}} \right\},$$

where $X^{c,\pi}(\Delta t) = r\Delta t(1-\pi)x + R\pi x$ and R is a random variable such that $E[R] = \alpha\Delta t$ and $Var[R] = \sigma^2\Delta t$. Then we have

$$\begin{aligned} & E_{0,x}\left[X^{\pi}(\Delta t)\right] - \psi(0,x)\left(Var_{0,x}\left[X^{\pi}(\Delta t)\right]\right)^{\frac{1}{2}} \\ &= r\Delta t x + (\alpha - r)\pi x\Delta t - \psi(0,x)\left(\sigma^2\Delta t\pi^2 x^2\right)^{\frac{1}{2}} \\ &= r\Delta t x + (\alpha - r)\pi x\Delta t - \psi(0,x)\sigma\sqrt{\Delta t}|\pi|x. \end{aligned}$$

We directly obtain the optimal strategy

$$\pi^* = \begin{cases} 0, & \text{if } (\alpha - r)\Delta t < \psi(0,x)\sigma\sqrt{\Delta t}, \\ \text{non-defined}(\mathbb{R}_+), & \text{if } (\alpha - r)\Delta t = \psi(0,x)\sigma\sqrt{\Delta t}, \\ \infty, & \text{if } (\alpha - r)\Delta t > \psi(0,x)\sigma\sqrt{\Delta t}. \end{cases}$$

Clearly, for any risk aversion function $\psi(0,x)$ there exist a small enough Δt such that $(\alpha - r)\Delta t < \psi(0,x)\sigma\sqrt{\Delta t}$, i.e. such that $\pi^* = 0$ [13]. Likewise, for a multi period problem we conclude that for any risk aversion function $\psi(t,x)$, fulfilling our reasonable assumptions, we get the Nash equilibrium strategy $\pi^* = 0$ whenever the discretization of the interval of optimization is fine enough. To obtain this note, by the argumentation above, that for a fine enough discretization of the interval player n (the last player in the game) chooses to take no risk at all. Consequently, player number $n-1$ faces, since there is no randomness after time T_{n-1}, also a single period problem and by the same argumentation player number $n-1$ also chooses to take no risk at all. Proceeding recursively we obtain $\pi^* = 0$ for all players.

For any risk aversion function $\psi(t,x)$, fulfilling our reasonable assumptions, we now conclude that over an infinitesimal time interval, dt, standard deviation is of the order $\sqrt{dt}$, which means that the punishment is so hard that any risk taking is unattractive.

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A Appendix

A.1

Proof of Theorem 2.1. Consider an arbitrary admissible strategy (c, π).

1: First we argue that if there exist a function Y such that, $\forall (t, x) \in (0, T) \times \mathbb{R}$, we have

$$Y_t = -[(r + \pi(\alpha - r))x + \ell - c]Y_x - \frac{1}{2}\pi^2\sigma^2x^2Y_{xx} - c + \rho Y, \tag{63}$$

$$Y(T, x) = x, \tag{64}$$

then

$$Y(t, x) = y^{c,\pi}(t, x), \tag{65}$$

where $y^{c,\pi}$ is given by (14).

To show this define

$$\widetilde{Y}(t, x) = e^{-\rho t}Y(t, x). \tag{66}$$

From (63) and (64) we get, $\forall (t, x) \in (0, T) \times \mathbb{R}$, that

$$\widetilde{Y}_t = -[(r + \pi(\alpha - r))x + \ell - c]\widetilde{Y}_x - \frac{1}{2}\pi^2\sigma^2x^2\widetilde{Y}_{xx} - e^{-\rho t}c, \tag{67}$$

$$\widetilde{Y}(T, x) = e^{-\rho T}x. \tag{68}$$

Using Itô's formula

$$\begin{aligned}
&\widetilde{Y}(t, X^{c,\pi}(t)) \\
= &-\int_t^T d\widetilde{Y}(s, X^{c,\pi}(s)) + \widetilde{Y}(T, X^{c,\pi}(T)) \\
= &-\int_t^T \Big(\widetilde{Y}_s(s, X^{c,\pi}(s)) + [(r + \pi(s)(\alpha - r))X^{c,\pi}(s) + \ell(s) - c(s)]\,\widetilde{Y}_x(s, X^{c,\pi}(s)) \\
&+ \frac{1}{2}\pi(s)^2\sigma^2X^{c,\pi}(s)^2\widetilde{Y}_{xx}(s, X^{c,\pi}(s))\Big)ds \\
&- \int_t^T \pi(s)\sigma X^{c,\pi}(s)\widetilde{Y}_x(s, X^{c,\pi}(s))dW(s) + \widetilde{Y}(T, X^{c,\pi}(T)).
\end{aligned}$$

Plug in (67) and (68) to obtain

$$\widetilde{Y}(t, X^{c,\pi}(t)) = \int_t^T e^{-\rho s} c(s) ds + e^{-\rho T} X^{c,\pi}(T) - \int_t^T \pi(s)\sigma X^{c,\pi}(s) \widetilde{Y}_x(s, X^{c,\pi}(s)) dW(s). \quad (69)$$

Since (c, π) is an admissible strategy taking the conditional expectation given $X(t) = x$ on both sides of the equality leaves us with

$$\widetilde{Y}(t, x) = E\left[\int_t^T e^{-\rho s} c(s) ds + e^{-\rho T} X^{c,\pi}(T) \,\middle|\, X(t) = x \right].$$

It is now clear that

$$Y(t, x) = e^{\rho t} \widetilde{Y}(t, x) = y^{c,\pi}(t, x).$$

2: Second we argue that if there exist a function Z such that, $\forall (t, x) \in (0, T) \times \mathbb{R}$, we have

$$Z_t = -[(r + \pi(\alpha - r))x + \ell - c] Z_x - \frac{1}{2}\pi^2\sigma^2 x^2 Z_{xx} - 2cY + 2\rho Z, \quad (70)$$

$$Z(T, x) = x^2, \quad (71)$$

then

$$Z(t, x) = z^{c,\pi}(t, x), \quad (72)$$

where $z^{c,\pi}$ is given by (15).

To show this define $\widetilde{Z}(t, x) = e^{-2\rho t} Z(t, x)$. From (70) and (71) we get, $\forall (t, x) \in (0, T) \times \mathbb{R}$, that

$$\widetilde{Z}_t = -[(r + \pi(\alpha - r))x + \ell - c] \widetilde{Z}_x - \frac{1}{2}\pi^2\sigma^2 x^2 \widetilde{Z}_{xx} - 2e^{-\rho t} c \widetilde{Y}, \quad (73)$$

$$\widetilde{Z}(T, x) = \left(e^{-\rho T} x\right)^2, \quad (74)$$

where $\widetilde{Y}(t, x)$ is given in (66). Using Itô's formula

$$\begin{aligned}
&\widetilde{Z}(t, X^{c,\pi}(t)) \\
&= -\int_t^T d\widetilde{Z}(s, X^{c,\pi}(s)) + \widetilde{Z}(T, X^{c,\pi}(T)) \\
&= -\int_t^T \Big(\widetilde{Z}_s(s, X^{c,\pi}(s)) + [(r + \pi(s)(\alpha - r))X^{c,\pi}(s) + \ell(s) - c(s)]\, \widetilde{Z}_x(s, X^{c,\pi}(s)) \\
&\quad + \frac{1}{2}\pi(s)^2\sigma^2 X^{c,\pi}(s)^2 \widetilde{Z}_{xx}(s, X^{c,\pi}(s)) \Big) ds \\
&\quad - \int_t^T \pi(s)\sigma X^{c,\pi}(s) \widetilde{Z}_x(s, X^{c,\pi}(s)) dW(s) + \widetilde{Z}(T, X^{c,\pi}(T)).
\end{aligned}$$

Plug in (73) and (74) and thereafter (69) to obtain

$$
\begin{aligned}
&\widetilde{Z}(t, X^{c,\pi}(t)) \\
&= 2\int_t^T e^{-\rho s} c(s) \widetilde{Y}(s, X^{c,\pi}(s)) ds + \left(e^{-\rho T} X^{c,\pi}(T)\right)^2 \qquad (75)\\
&\quad - \int_t^T \pi(s)\sigma X^{c,\pi}(s) \widetilde{Z}_x(s, X^{c,\pi}(s)) dW(s) \\
&= 2\int_t^T e^{-\rho s} c(s) \int_s^T e^{-\rho y} c(y) dy ds + 2\int_t^T e^{-\rho s} c(s) ds\, e^{-\rho T} X^{c,\pi}(T) \\
&\quad - 2\int_t^T e^{-\rho s} c(s) \int_s^T \pi(y)\sigma X^{c,\pi}(y) \widetilde{Y}_x(y, X^{c,\pi}(y)) dW(y) ds \\
&\quad + \left(e^{-\rho T} X^{c,\pi}(T)\right)^2 - \int_t^T \pi(s)\sigma X^{c,\pi}(s) \widetilde{Z}_x(s, X^{c,\pi}(s)) dW(s). \qquad (76)
\end{aligned}
$$

Since (c, π) is an admissible strategy taking the conditional expectation given $X(t) = x$ on both sides of the equality leaves us with

$$
\begin{aligned}
\widetilde{Z}(t, x) = E\Bigg[&2\int_t^T e^{-\rho s} c(s) \int_s^T e^{-\rho y} c(y) dy ds \\
&+ 2\int_t^T e^{-\rho s} c(s) ds\, e^{-\rho T} X^{c,\pi}(T) + \left(e^{-\rho T} X^{c,\pi}(T)\right)^2 \Bigg| X(t) = x \Bigg].
\end{aligned}
$$

Provided that $2\int_t^T e^{-\rho s} c(s) \int_s^T e^{-\rho y} c(y) dy ds = \left(\int_t^T e^{-\rho s} c(s) ds\right)^2$ we now have that

$$
\widetilde{Z}(t, x) = E\left[\left(\int_t^T e^{-\rho s} c(s) ds + e^{-\rho T} X^{c,\pi}(T) \right)^2 \Bigg| X(t) = x \right].
$$

This is however easily realized since

$$
\frac{\partial}{\partial t}\left(2\int_t^T e^{-\rho s} c(s) \int_s^T e^{-\rho y} c(y) dy ds\right) = -2e^{-\rho t} c(t) \int_t^T e^{-\rho y} c(y) dy = \frac{\partial}{\partial t}\left(\int_t^T e^{-\rho s} c(s) ds\right)^2
$$

$$
\left(2\int_t^T e^{-\rho s} c(s) \int_s^T e^{-\rho y} c(y) dy ds\right)_{\big| t=T} = 0 = \left(\int_t^T e^{-\rho s} c(s) ds\right)^2_{\big| t=T}.
$$

It is now clear that

$$
Z(t, x) = e^{2\rho t} \widetilde{Z}(t, x) = z^{c,\pi}(t, x).
$$

3: At last we argue that if there exist a function F such that, $\forall (t, x) \in (0, T) \times \mathbb{R}$, we have

$$
F_t = \inf_{(c,\pi)\in\mathcal{A}} \left\{ -[(r + \pi(\alpha - r))x + \ell - c](F_x - Q) - \frac{1}{2}\pi^2\sigma^2 x^2 (F_{xx} - U) + J \right\}, \qquad (77)
$$

$$
F(T, x) = fc, \pi\left(T, x, x, x^2\right), \qquad (78)
$$

where

$$
Q = f_x^{c^*,\pi^*}, \qquad (79)
$$

$$
U = f_{xx}^{c^*,\pi^*} + \left(F_x^{(1)}\right)^2 f_{yy}^{c^*,\pi^*} + \left(F_x^{(2)}\right)^2 f_{zz}^{c^*,\pi^*} + 2F_x^{(1)} F_x^{(2)} f_{yz}^{c^*,\pi^*} + 2F_x^{(1)} f_{xy}^{c^*,\pi^*} + 2F_x^{(2)} f_{xz}^{c^*,\pi^*}, \qquad (80)
$$

$$
J = \left(\rho F^{(1)} - c\right) f_y^{c^*,\pi^*} + 2\left(\rho F^{(2)} - cF^{(1)}\right) f_z^{c^*,\pi^*} + f_t^{c^*,\pi^*}, \qquad (81)
$$

with $F^{(1)}$ and $F^{(2)}$ fulfilling (17) and (18), respectively, then

$$F(t,x) = V(t,x),$$

where V is the optimal value function defined by (11).

First step is to derive an expression for

$$f^{c,\pi}(t, X^{c,\pi}(t), y^{c,\pi}(t, X^{c,\pi}(t)), z^{c,\pi}(t, X^{c,\pi}(t))).$$

By (65) and (72) this equals

$$f^{c,\pi}(t, X^{c,\pi}(t), Y(t, X^{c,\pi}(t)), Z(t, X^{c,\pi}(t))).$$

Using Itô's formula $\big($we have assumed $f \in \mathcal{C}^{1,2,2,2}\big)$ we get that

$$\begin{aligned}
& f^{c,\pi}(t, X^{c,\pi}(t), y^{c,\pi}(t, X^{c,\pi}(t)), z^{c,\pi}(t, X^{c,\pi}(t))) \\
= & -\int_t^T df^{c,\pi}(s, X^{c,\pi}(s), Y(s, X^{c,\pi}(s)), Z(s, X^{c,\pi}(s))) \\
& + f^{c,\pi}(T, X^{c,\pi}(T), Y(T, X^{c,\pi}(T)), Z(T, X^{c,\pi}(T))) \\
= & -\int_t^T \Big\{ (f_s^{c,\pi} + f_y^{c,\pi} Y_s + f_z^{c,\pi} Z_s) ds + (f_x^{c,\pi} + f_y^{c,\pi} Y_x + f_z^{c,\pi} Z_x) dX^{c,\pi}(s) \\
& + \frac{1}{2}\Big[f_y^{c,\pi} Y_{xx} + f_z^{c,\pi} Z_{xx} + f_{xx}^{c,\pi} + f_{yy}^{c,\pi} (Y_x)^2 + f_{zz}^{c,\pi} (Z_x)^2 \\
& + 2 f_{xy}^{c,\pi} Y_x + 2 f_{xz}^{c,\pi} Z_x + 2 f_{yz}^{c,\pi} Y_x Z_x \Big] \pi(s)^2 \sigma^2 X^{c,\pi}(s)^2 ds \Big\} \\
& + f^{c,\pi}(T, X^{c,\pi}(T), Y(T, X^{c,\pi}(T)), Z(T, X^{c,\pi}(T))),
\end{aligned} \tag{82}$$

where we have skipped some arguments under the integral. Inserting (63), (70) and the dynamics of X given by (5) we have that

$$\begin{aligned}
& f^{c,\pi}(t, X^{c,\pi}(t), y^{c,\pi}(t, X^{c,\pi}(t)), z^{c,\pi}(t, X^{c,\pi}(t))) \\
= & -\int_t^T \Big\{ \Big[f_s^{c,\pi} + f_y^{c,\pi} \Big(-\big[(r + \pi(s)(\alpha - r)) X^{c,\pi}(s) + \ell(s) - c(s) \big] Y_x \\
& - \frac{1}{2}\pi(s)^2 \sigma^2 X^{c,\pi}(s)^2 Y_{xx} - c(s) + \rho Y \Big) + f_z^{c,\pi} \Big(-\big[(r + \pi(s)(\alpha - r)) X^{c,\pi}(s) + \ell(s) - c(s) \big] Z_x \\
& - \frac{1}{2}\pi(s)^2 \sigma^2 X^{c,\pi}(s)^2 Z_{xx} - 2c(s) Y + 2\rho Z \Big) \Big] ds \\
& + (f_x^{c,\pi} + f_y^{c,\pi} Y_x + f_z^{c,\pi} Z_x) \Big(\big[(r + \pi(s)(\alpha - r)) X^{c,\pi}(s) + \ell(s) - c(s) \big] ds + \pi(s) \sigma X^{c,\pi}(s) dW(s) \Big) \\
& + \frac{1}{2}\Big[f_y^{c,\pi} Y_{xx} + f_z^{c,\pi} Z_{xx} + f_{xx}^{c,\pi} + f_{yy}^{c,\pi} (Y_x)^2 + f_{zz}^{c,\pi} (Z_x)^2 \\
& + 2 f_{xy}^{c,\pi} Y_x + 2 f_{xz}^{c,\pi} Z_x + 2 f_{yz}^{c,\pi} Y_x Z_x \Big] \pi(s)^2 \sigma^2 X^{c,\pi}(s)^2 ds \Big\} \\
& + f^{c,\pi}(T, X^{c,\pi}(T), Y(T, X^{c,\pi}(T)), Z(T, X^{c,\pi}(T))).
\end{aligned}$$

After reduction this is

$$
\begin{aligned}
& f^{c,\pi}(t, X^{c,\pi}(t), y^{c,\pi}(t, X^{c,\pi}(t)), z^{c,\pi}(t, X^{c,\pi}(t))) \\
= & -\int_t^T \bigg\{ \Big(f_s^{c,\pi} + f_y^{c,\pi}\big(-c(s) + \rho Y \big) + f_z^{c,\pi}\big(-2c(s)Y + 2\rho Z \big) \Big) ds \\
& + f_x^{c,\pi}\big[(r + \pi(s)(\alpha - r))X^{c,\pi}(s) + \ell(s) - c(s)\big] ds + (f_x^{c,\pi} + f_y^{c,\pi} Y_x + f_z^{c,\pi} Z_x)\pi(s)\sigma X^{c,\pi}(s) dW(s) \\
& + \frac{1}{2}\Big[f_{xx}^{c,\pi} + f_{yy}^{c,\pi}(Y_x)^2 + f_{zz}^{c,\pi}(Z_x)^2 + 2 f_{xy}^{c,\pi} Y_x + 2 f_{xz}^{c,\pi} Z_x + 2 f_{yz}^{c,\pi} Y_x Z_x \Big] \pi(s)^2 \sigma^2 X^{c,\pi}(s)^2 ds \bigg\} \\
& + f^{c,\pi}(T, X^{c,\pi}(T), Y(T, X^{c,\pi}(T)), Z(T, X^{c,\pi}(T))).
\end{aligned}
$$

Define for an arbitrary admissible strategy (c, π) the quantities corresponding to (80) and (81) by

$$
\widetilde{U} = f_{xx}^{c,\pi} + (Y_x)^2 f_{yy}^{c,\pi} + (Z_x)^2 f_{zz}^{c,\pi} + 2Y_x Z_x f_{yz}^{c,\pi} + 2Y_x f_{xy}^{c,\pi} + 2Z_x f_{xz}^{c,\pi}, \tag{83}
$$

$$
\widetilde{J} = (\rho Y - c) f_y^{c,\pi} + 2(\rho Z - cY) f_z^{c,\pi} + f_t^{c,\pi}. \tag{84}
$$

We now get

$$
\begin{aligned}
& f^{c,\pi}(t, X^{c,\pi}(t), y^{c,\pi}(t, X^{c,\pi}(t)), z^{c,\pi}(t, X^{c,\pi}(t))) \\
= & -\int_t^T \Big\{ \widetilde{J}(s) + f_x^{c,\pi}\big[(r + \pi(s)(\alpha - r))X^{c,\pi}(s) + \ell(s) - c(s)\big] ds \\
& + (f_x^{c,\pi} + f_y^{c,\pi} Y_x + f_z^{c,\pi} Z_x)\pi(s)\sigma X^{c,\pi}(s) dW(s) + \frac{1}{2}\widetilde{U}(s)\pi(s)^2\sigma^2 X^{c,\pi}(s)^2 ds \Big\} \\
& + f^{c,\pi}(T, X^{c,\pi}(T), Y(T, X^{c,\pi}(T)), Z(T, X^{c,\pi}(T))).
\end{aligned} \tag{85}
$$

Next step is, by use of (85), to show that for any admissible strategy (c, π) we have

$$
\begin{aligned}
& f(t, x, y^{c,\pi}(t, x), z^{c,\pi}(t, x)) \\
\leq\ & F(t, x) + \int_t^T \bigg\{ [(r + \pi(s)(\alpha - r))X^{c,\pi}(s) + \ell(s) - c(s)] \Big(f_x^{c^*,\pi^*}(s) - f_x^{c,\pi}(s) \Big) \\
& \qquad + J(s) - \widetilde{J}(s) + \frac{1}{2}\sigma^2 \pi(s)^2 X^{c,\pi}(s)^2 (U(s) - \widetilde{U}(s)) \bigg\} ds.
\end{aligned} \tag{86}
$$

By use of Itô's formula we get that

$$
\begin{aligned}
F(t, X^{c,\pi}(t)) &= -\int_t^T dF(s, X^{c,\pi}(s)) + F(T, X^{c,\pi}(T)) \\
&= -\int_t^T \Big(F_s ds + F_x dX^{c,\pi}(s) + \frac{1}{2} F_{xx}\pi(s)^2\sigma^2 X^{c,\pi}(s)^2 ds \Big) \\
&\quad + F(T, X^{c,\pi}(T)).
\end{aligned}
$$

Since F solves the pseudo Hamilton-Jacobi-Bellman equation (77) we see that for the arbitrary strategy (c, π) we have, $\forall (t, x) \in (0, T) \times \mathbb{R}$, that

$$
F_t \leq -\big[(r + \pi(\alpha - r))x + \ell - c\big](F_x - Q) - \frac{1}{2}\pi^2\sigma^2 x^2 (F_{xx} - U) + J.
$$

Inserting this with $x = X^{c,\pi}(s)$, inserting the dynamics of X given by (5), and inserting the terminal conditions (78), (64) and (71) we get that

$$
\begin{aligned}
F(t, X^{c,\pi}(t)) \geq\ & - \int_t^T \bigg\{ \Big(- \big[(r + \pi(s)(\alpha - r))X^{c,\pi}(s) + \ell(s) - c(s)\big](F_x - Q) \\
& - \frac{1}{2}\pi(s)^2\sigma^2 X^{c,\pi}(s)^2(F_{xx} - U(s)) + J(s)\Big) ds \\
& + F_x \Big(\big[(r + \pi(s)(\alpha - r))X^{c,\pi}(s) + \ell(s) - c(s)\big] ds + \pi(s)\sigma X^{c,\pi}(s) dW(s)\Big) \\
& + \frac{1}{2} F_{xx}\pi(s)^2\sigma^2 X^{c,\pi}(s)^2 ds \bigg\} + f^{c,\pi}(T, X^{c,\pi}(T), Y(T, X^{c,\pi}(T)), Z(T, X^{c,\pi}(T))).
\end{aligned}
$$

After reduction this is (remember $Q = f_x^{c^*,\pi^*}$)

$$
\begin{aligned}
F(t, X^{c,\pi}(t)) \geq\ & - \int_t^T \bigg\{ \Big(\big[(r + \pi(s)(\alpha - r))X^{c,\pi}(s) + \ell(s) - c(s)\big] f_x^{c^*,\pi^*}(s) \\
& + \frac{1}{2}\pi(s)^2\sigma^2 X^{c,\pi}(s)^2 U(s) + J(s)\Big) ds + F_x \pi(s)\sigma X^{c,\pi}(s) dW(s) \bigg\} \\
& + f(T, X^{c,\pi}(T), Y(T, X^{c,\pi}(T)), Z(T, X^{c,\pi}(T))).
\end{aligned}
$$

Insert (85) to obtain

$$
\begin{aligned}
F(t, X^{c,\pi}(t)) \geq\ & - \int_t^T \bigg\{ [(r + \pi(s)(\alpha - r))X^{c,\pi}(s) + \ell(s) - c(s)] \Big(f_x^{c^*,\pi^*}(s) - f_x^{c,\pi}(s) \Big) \\
& \qquad + J(s) - \widetilde{J}(s) + \frac{1}{2}\sigma^2\pi(s)^2 X^{c,\pi}(s)^2 (U(s) - \widetilde{U}(s)) \bigg\} ds \\
& + \int_t^T \big(f_x^{c,\pi} + f_y^{c,\pi} Y_x + f_z^{c,\pi} Z_x - F_x \big) \pi(s)\sigma X^{c,\pi}(s) dW(s) \\
& + f^{c,\pi}(t, X^{c,\pi}(t), y^{c,\pi}(t, X^{c,\pi}(t)), z^{c,\pi}(t, X^{c,\pi}(t)))
\end{aligned}
\tag{87}
$$

Since (c, π) is an arbitrarily chosen admissible strategy, taking the conditional expectation given $X(t) = x$ of the inequality gives (86).

Consider now the specific strategy (c^*, π^*) fulfilling the infimum in (77). By (65) and (72) it follows that

$$
\begin{aligned}
F^{(1)}(t, x) &= y^{c^*,\pi^*}(t, x), \\
F^{(2)}(t, x) &= z^{c^*,\pi^*}(t, x).
\end{aligned}
$$

Final step is to show that the Nash equilibrium criteria given by Definition 2.1 is fulfilled.

Going through the same calculations as above, and using that for the specific strategy (c^*, π^*) we have, $\forall (t,x) \in (0,T) \times \mathbb{R}$, that

$$F_t = -[(r + \pi^*(\alpha - r))x + \ell - c^*](F_x - Q) - \frac{1}{2}(\pi^*)^2\sigma^2 x^2 (F_{xx} - U) + J,$$

we get

$$\begin{aligned} F\left(t, X^{c^*,\pi^*}(t)\right) = & \int_t^T \left\{\left(f_x^{c^*,\pi^*} + f_y^{c^*,\pi^*} Y_x + f_z^{c^*,\pi^*} Z_x - F_x\right)\pi^*(s)\sigma X^{c^*,\pi^*}(s) dW(s)\right\} \\ & + f^{c^*,\pi^*}\left(t, X^{c^*,\pi^*}(t), Y\left(t, X^{c^*,\pi^*}(t)\right), Z\left(t, X^{c^*,\pi^*}(t)\right)\right). \end{aligned} \tag{88}$$

Since (c^*, π^*) is an admissible strategy, taking the conditional expectation given $X(t) = x$ on both sides of the inequality we obtain

$$F(t,x) = f^{c^*,\pi^*}\left(t, x, y^{c^*,\pi^*}(t,x), y^{c^*,\pi^*}(t,x)\right). \tag{89}$$

Now let $(\widetilde{c}_h, \widetilde{\pi}_h)$ be defined by (10). To make it explicit that the expressions in (83) and (84) depend on h, when using the strategy $(\widetilde{c}_h, \widetilde{\pi}_h)$, we write $\widetilde{U}_h$ and $\widetilde{J}_h$, respectively. By (86) and (89) we get that

$$\begin{aligned} & \liminf_{h\to 0} \frac{f^{c^*,\pi^*}\left(t, x, y^{c^*,\pi^*}(t,x), z^{c^*,\pi^*}(t,x)\right) - f^{\widetilde{c}_h,\widetilde{\pi}_h}\left(t, x, y^{\widetilde{c}_h,\widetilde{\pi}_h}(t,x), z^{\widetilde{c}_h,\widetilde{\pi}_h}(t,x)\right)}{h} \\ \geq & \liminf_{h\to 0} \left\{ \frac{\int_t^T \left[(r + \widetilde{\pi}_h(s)(\alpha - r))X^{\widetilde{c}_h,\widetilde{\pi}_h}(s) + \ell(s) - \widetilde{c}_h(s)\right]\left(f_x^{\widetilde{c}_h,\widetilde{\pi}_h}(s) - f_x^{c^*,\pi^*}(s)\right) ds}{h} \right. \\ & \left. + \frac{\int_t^T \left(\widetilde{J}_h(s) - J(s) + \frac{1}{2}\sigma^2 \widetilde{\pi}_h(s)^2 X^{\widetilde{c}_h,\widetilde{\pi}_h}(s)^2 (\widetilde{U}_h(s) - U(s))\right) ds}{h} \right\} \\ = & \liminf_{h\to 0} \left\{ \frac{\int_t^{t+h} [(r + \pi(s)(\alpha - r))X^{c,\pi}(s) + \ell(s) - c(s)]\left(f_x^{\widetilde{c}_h,\widetilde{\pi}_h}(s) - f_x^{c^*,\pi^*}(s)\right) ds}{h} \right. \\ & \left. + \frac{\int_t^{t+h} \left(\widetilde{J}_h(s) - J(s) + \frac{1}{2}\sigma^2 \pi(s)^2 X^{c,\pi}(s)^2 (\widetilde{U}_h(s) - U(s))\right) ds}{h} \right\} \\ = & [(r + \pi(t)(\alpha - r))X^{c,\pi}(t) + \ell(t) - c(t)]\left(f_x^{\widetilde{c}_0,\widetilde{\pi}_0}(t) - f_x^{c^*,\pi^*}(t)\right) \\ & + \widetilde{J}_0(t) - J(t) + \frac{1}{2}\sigma^2 \pi(t)^2 X^{c,\pi}(t)^2 (\widetilde{U}_0(t) - U(t)) \\ = & 0, \end{aligned}$$

where we have used that $(\widetilde{c}_h, \widetilde{\pi}_h)$ coincides with (c^*, π^*) on $[t+h, T]$, and with (c, π) on [t,t+h). We conclude that $F(t,x) = V(t,x)$, and that (c^*, π^*) is the corresponding optimal strategy.

□

A.2

First. We want to show that the assumption made in (44) given as

$$a(t)b(t) = \frac{g(t)}{2}$$

is fulfilled. This is easily verified since $a(T)b(T) = \frac{g(T)}{2} = 0$, and by use of (52) and (53)

$$\begin{aligned}
&\frac{\partial}{\partial t}(a(t)b(t)) = a'(t)b + ab'(t) \\
&= -\left\{(r-\rho) + \frac{(\alpha-r)^2}{\sigma^2\gamma f(t)}\left[a(t) + \gamma(a(t)^2 - f(t))\right]\right\} a(t)b(t) - c^*(t)a(t) + \rho a(t)b(t),
\end{aligned}$$

and

$$\frac{g'(t)}{2} = -\left\{(r-\rho) + \frac{(\alpha-r)^2}{\sigma^2\gamma f(t)}\left[a(t) + \gamma(a(t)^2 - f(t))\right]\right\}\frac{g(t)}{2} - c^*(t)a(t) + \rho\frac{g(t)}{2}.$$

□

Second. We want to show that the condition (51) for the non-binding case given as

$$\frac{\gamma f(t)}{x + K^{(c^*)}(t)} > 0, \ \forall (t,x) \in [0,T) \times \mathbb{R},$$

required for the investment strategy (50) to be optimal, is fulfilled.

In order to verify this condition notice that by use of the candidate for the optimal strategy given by (49) and (50) we get that

$$\begin{aligned}
&d\left(X^{c^*,\pi^*}(t) + K^{(c^*)}(t)\right) \\
&= \left[rX^{c^*,\pi^*}(t) + \frac{(\alpha-r)^2}{\sigma^2\gamma f(t)}\left[a(t) + \gamma\left(a(t)^2 - f(t)\right)\right]\left(X^{c^*,\pi^*}(t) + K^{(c^*)}(t)\right) + \ell(t) - c^*(t)\right] dt \\
&\quad + \frac{\alpha-r}{\sigma\gamma f(t)}\left[a(t) + \gamma\left(a(t)^2 - f(t)\right)\right]\left(X^{c^*,\pi^*}(t) + K^{(c^*)}(t)\right) dW(t) + \left(rK^{(c^*)}(t) - \ell(t) + c^*(t)\right) dt \\
&= \left(r + \frac{(\alpha-r)^2}{\sigma^2\gamma f(t)}\left[a(t) + \gamma\left(a(t)^2 - f(t)\right)\right]\right)\left(X^{c^*,\pi^*}(t) + K^{(c^*)}(t)\right) dt \\
&\quad + \frac{\alpha-r}{\sigma\gamma f(t)}\left[a(t) + \gamma\left(a(t)^2 - f(t)\right)\right]\left(X^{c^*,\pi^*}(t) + K^{(c^*)}(t)\right) dW(t).
\end{aligned}$$

We see that $(X^{c^*,\pi^*}(t) + K^{(c^*)}(t))$ takes the form of a geometric Brownian motion. We get the solution

$$\begin{aligned}
&X^{c^*,\pi^*}(t) + K^{(c^*)}(t) \\
&= \left(x_0 + K^{(c^*)}(0,x_0)\right)\exp\left\{\int_0^t \left(r + \frac{(\alpha-r)^2}{\sigma^2\gamma f(s)}\left[a(s) + \gamma\left(a(s)^2 - f(s)\right)\right]\right.\right. \\
&\quad \left.\left. - \frac{1}{2}\frac{(\alpha-r)^2}{\sigma^2\gamma^2 f(s)^2}\left[a(s) + \gamma\left(a(s)^2 - f(s)\right)\right]^2\right) ds + \int_0^t \frac{\alpha-r}{\sigma\gamma f(s)}\left[a(s) + \gamma\left(a(s)^2 - f(s)\right)\right] dW(s)\right\}.
\end{aligned} \tag{90}$$

From (95) below it follows that f is a strictly positive function over the time interval $[0,T)$. Finally, since for the non-binding case $x_0 + K^{(c^*)}(0,x_0) > 0$ we conclude that the condition is fulfilled. □

Third. We want to show that the highly non-linear system of partial differential equations given by (52) and (53) has a unique global solution. In order to show this take conditional expectation in (90) for $t = T$ to get that (remember that $K^{(c^*)}(T) = 0$)

$$E_{t,x}\left[X^{c^*,\pi^*}(T)\right] = \left(x + K^{(c^*)}(t)\right) e^{\int_t^T [r+(\alpha-r)\widetilde{\pi}^*(s)]ds} \tag{91}$$

$$E_{t,x}\left[\left(X^{c^*,\pi^*}(T)\right)^2\right] = \left(x + K^{(c^*)}(t)\right)^2 e^{2\int_t^T \left[r+(\alpha-r)\widetilde{\pi}^*(s)+\frac{1}{2}\sigma^2\widetilde{\pi}^*(s)^2\right]ds}, \tag{92}$$

where[14]

$$\widetilde{\pi}^*(t) = \frac{\alpha - r}{\sigma^2\gamma f(t)}\left[a(t) + \gamma\left(a(t)^2 - f(t)\right)\right]. \tag{93}$$

We now get that

$$\begin{aligned}
& a(t)\left(x + K^{(c^*)}(t)\right) + b(t) \\
&= E_{t,x}\left[\int_t^T e^{-\rho(s-t)}c^*(s)ds + e^{-\rho(T-t)}X^{c^*,\pi^*}(T)\right] \\
&= \left(x + K^{(c^*)}(t)\right) e^{\int_t^T [(r-\rho)+(\alpha-r)\widetilde{\pi}^*(s)]ds} + \int_t^T e^{-\rho(s-t)}c^*(s)ds
\end{aligned}$$

and

$$\begin{aligned}
& f(t)\left(x + K^{(c^*)}(t)\right)^2 + g(t)\left(x + K^{(c^*)}(t)\right) + h(t) \\
&= E_{t,x}\left[\left(\int_t^T e^{-\rho(s-t)}c^*(s)ds + e^{-\rho(T-t)}X^{c^*,\pi^*}(T)\right)^2\right] \\
&= e^{-2\rho(T-t)}E_{t,x}\left[\left(X^{c^*,\pi^*}(T)\right)^2\right] + 2\left(\int_t^T e^{-\rho(s-t)}c^*(s)ds\right)e^{-\rho(T-t)}E_{t,x}\left[X^{c^*,\pi^*}(T)\right] \\
&\quad + \left(\int_t^T e^{-\rho(s-t)}c^*(s)ds\right)^2 \\
&= \left(x + K^{(c^*)}(t)\right)^2 e^{2\int_t^T \left[(r-\rho)+(\alpha-r)\widetilde{\pi}^*(s)+\frac{1}{2}\sigma^2\widetilde{\pi}^*(s)^2\right]ds} \\
&\quad + 2\left(\int_t^T e^{-\rho(s-t)}c^*(s)ds\right)\left(x + K^{(c^*)}(t)\right) e^{\int_t^T [(r-\rho)+(\alpha-r)\widetilde{\pi}^*(s)]ds} \\
&\quad + \left(\int_t^T e^{-\rho(s-t)}c^*(s)ds\right)^2.
\end{aligned}$$

[14] Notice that $\widetilde{\pi}$ defines the optimal proposition of $X^{c^*,\pi^*} + K^{(c^*)}$ to invest in stocks S.

Collecting terms we obtain

$$
\begin{aligned}
a(t) &= e^{\int_t^T [(r-\rho)+(\alpha-r)\widetilde{\pi}^*(s)]ds}, &(94)\\
b(t) &= \int_t^T e^{-\rho(s-t)} c^*(s)ds,\\
f(t) &= e^{2\int_t^T \left[(r-\rho)+(\alpha-r)\widetilde{\pi}^*(s)+\frac{1}{2}\sigma^2\widetilde{\pi}^*(s)^2\right]ds}, &(95)\\
g(t) &= 2e^{\int_t^T [(r-\rho)+(\alpha-r)\widetilde{\pi}^*(s)]ds} \int_t^T e^{-\rho(s-t)} c^*(s)ds,\\
h(t) &= \left(\int_t^T e^{-\rho(s-t)} c^*(s)ds\right)^2.
\end{aligned}
$$

Now insert (94) and (95) into (93) to get the following integral equation for $\widetilde{\pi}^*$

$$
\begin{aligned}
\widetilde{\pi}^*(t) &= \frac{\alpha-r}{\sigma^2\gamma f(t)}\left[a(t)+\gamma\left(a(t)^2-f(t)\right)\right]\\
&= \frac{\alpha-r}{\sigma^2\gamma}\left\{e^{-\int_t^T\left[(r-\rho)+(\alpha-r)\widetilde{\pi}^*(s)+\sigma^2\widetilde{\pi}^*(s)^2\right]ds}+\gamma e^{-\int_t^T\sigma^2\widetilde{\pi}^*(s)^2ds}-\gamma\right\}. &(96)
\end{aligned}
$$

The key question now is whether the integral equation (96) has a unique global solution. Fortunately this is the case. The technical work is done in Björk et al. (2012). They show that the algorithm given by

$$
\begin{aligned}
\widetilde{\pi}_0(t) &:= 1,\\
\widetilde{\pi}_{n+1}(t) &= \frac{\alpha-r}{\sigma^2\gamma}\left[e^{-\int_t^T[(r-\rho)+(\alpha-r)\widetilde{\pi}_n(s)+\sigma^2\widetilde{\pi}_n^2(s)]ds}+\gamma e^{-\int_t^T\sigma^2\widetilde{\pi}_n^2(s)ds}-\gamma\right], &(97)
\end{aligned}
$$

converges in $\mathcal{C}[0,T]$, and that the full sequence $\{\widetilde{\pi}_n\}$ converges to the solution $\widetilde{\pi}^*$. □