

Entrance times of random walks: With applications to pension fund modeling

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Abstract

The purpose of the paper is twofold. First, we consider entrance times of random walks, i.e. the time of first entry to the negative axis. Partition sum formulas are given for entrance time probabilities, the n^{th} derivative of the generating function, and the n^{th} falling factorial entrance time moment. Similar formulas for the characteristic function of the position of the random walk both conditioned on entry and conditioned on no entry are also established. Second, we consider a model for a with-profits collective pension fund. The model has previously been studied by approximate methods, but we give here an essentially complete theoretical description of the model based on the entrance time results. We also conduct a mean-variance analysis for a fund in stationarity. To facilitate the analysis we devise a simple and effective exact simulation algorithm for sampling from the stationary distribution of a regenerative Markov chain.

Keywords: Random walks; entrance times; generating function; factorial moment; partition sum; stationarity; exact simulation; collective pension fund; with-profits contracts.

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1 Introduction

This paper analyzes a “with-profits collective pension scheme”; this type of scheme and variants hereof are widespread, among other places, in the Nordic countries and the Netherlands. Members of the scheme are guaranteed a minimum benefit. The guarantees are a liability for the pension fund for which it must reserve an amount of money equal to the net present value of future guaranteed benefits (the *reserve*). In addition to the already guaranteed benefits members may receive *bonuses* in the form of increased guarantees. Bonuses are attributed periodically, e.g. annually, when the ratio of total assets to the reserve (the *funding ratio*) is sufficiently high. The phrase “with-profits” refers to this profit-sharing mechanism.

Assets in excess of the reserve are termed the *bonus potential*. The bonus potential allows the fund to invest in risky assets by absorbing adverse investment results. The scheme is “collective” in the sense that the bonus potential is considered common to all members. It is also collective in the sense that the investment strategy and bonus policy is the same for all members. Collective funds generally benefit from economy of scale in the form of low administration and investment costs. The flip side is the lack of an individual investment strategy.

We consider a model for a collective pension fund in which a bonus is attributed when the funding ratio exceeds a given *bonus threshold*. The fund follows a CPPI (Constant Proportion Portfolio Insurance) investment strategy in order to stay solvent, i.e. to ensure that total assets exceed the reserve. The paper gives an essentially complete description of the fund dynamics including the time between bonuses, the (conditional) expected bonus percentage and the (conditional) expected funding ratio. The analysis is based on a detailed study of an embedded one-sided random walk obtained by a transformation of the funding ratio process sampled at the discrete set of time points where a bonus can be attributed. We consider both a fund starting at the bonus threshold and a fund in stationarity. Furthermore, we use the results to perform a mean-variance analysis of a standardized benefit payout. The analysis is performed for a fund in stationarity representing the “average” member. To facilitate the analysis we also employ an *exact simulation algorithm*, which might be of independent interest.

The theoretical foundation for the analysis is a series of new results for entrance times of random walks derived in this paper. For a random walk started at the origin the entrance time is the time of entry into $(-\infty, 0]$. The main theoretical results are partition sum formulas for the n^{th} derivative of the entrance time generating function and for the n^{th} falling factorial entrance time moment. The latter result generalizes the well-known formula for the mean entrance time. We also give a partition sum formula for entrance time probabilities, and a similar formula for the position of the random walk conditioned on entrance at time n . These results are implicit in Spitzer (1956) and Asmussen (2003), but the proofs are new and simpler. Finally, we give a new partition sum formula for the position of the random walk conditioned on entrance taken place after time n .

The results allow explicit calculations of the entrance time probabilities and moments in terms of the marginal distribution of the random walk. The computational effort gradually becomes prohibitive, but the first 100, say, entrance time probabilities and moments are computationally feasible. We use the results to study the one-sided random walk embedded in the funding ratio process by utilizing the fact that a one-sided random walk and its associated (unrestricted) random walk are identical up to the time of first entry into $(-\infty, 0]$. However, the results are generally applicable and not limited to our pension fund application.

Optimizing utility from terminal wealth for an individual saving for retirement is treated by numerous papers. The foundation was laid by Richard (1975) and to mention a few, who among other results obtain optimal investment strategies, there is, Huang and Milevsky (2008) who allow for unspanned labor income; Huang et al. (2008) who separate the breadwinner income process from the family consumption process; Steffensen and Kraft (2008) who generalize to a multi-state Markov chain framework typically used by actuaries for modeling a series of life history events; Bruhn and Steffensen (2011) who generalize to a multi-person household, with focus on a married couple with economically and/or probabilistically dependent members; Kwak et al.

(2011) who also consider a household but focus on generation issues; Kronborg and Steffensen (2013) who calculate the optimal investment strategy for a pension saver in the presence of a minimum rate guarantee; and Gerber and Shiu (2000) who present a comprehensive discussion of terminal utility optimization in a pension saving context. There is also a vast literature on modern investment management, founded by Markowitz (1952) and Merton (1971), aimed at finding optimal investment strategies without the pension aspect.

In contrast to the literature cited above this paper takes the point of view of a pension fund where a group of people share a common investment strategy. Investment gains are shared through a bonus strategy by which collective funds above a threshold are transferred to individual guarantees. Based on results on utility optimization of durable goods by Hindi and Huang (1993) it can be shown that the optimal bonus strategy is to continuously attribute bonuses whenever the funding ratio exceeds a certain barrier — thereby not allowing the funding ratio to exceed the barrier. In this paper, and in real life, the transfer is done periodically rather than continuously. References for continuous-time analysis of pension schemes taking both assumed (technical) returns, realized returns and bonus into account include Norberg (1999), Steffensen (2000), Norberg (2001), Steffensen (2004) and Nielsen (2006).

We assume the fund to follow a CPPI strategy. This strategy ensures that the fund remains funded, and it is “locally” optimal if we consider the periods between possible bonus attributions as local horizons. More precisely, Preisel et al. (2010) point out that CPPI is optimal in a finite horizon setting with HARA-utility and a subsistence level corresponding to a terminal funding ratio of one. CPPI strategies are treated, for the unrestricted case, by Cox and Huang (1989), and for the restricted case, by Teplá (2001). Using a CPPI strategy, and thereby avoiding insolvency, as done in this paper, stands in contrast to the literature on constructing contracts that are fair between owners and policyholders, see e.g. Briys and de Varenne (1997) and Grosen and Jørgensen (2000).

The current paper is related to Preisel et al. (2010), Kryger (2010) and Kryger (2011). We use the same underlying funding ratio dynamics, but the pension product and the terms by which members enter and leave the fund differ. In the cited papers members pay a fixed share (possibly zero) of contributions to the bonus potential on entry. This raises a number of issues regarding intergenerational fairness. In the present setup the share depends on the funding status of the fund in such a way that the contract is always financially fair. Methodologically, the cited papers use various analytical approximations while the current paper relies almost exclusively on exact results.

The main insight of Preisel et al. (2010) is that a given year’s apparent success of a large bonus resulting from a high equity allocation can come at the even higher price of subsequent large losses trapping the company at a low funding ratio for a long period. They also derive approximations to the expected bonus and funding ratio in stationarity. Kryger (2010) finds optimal investment strategies for power utility and mean-variance criteria. For fixed values of the bonus threshold, he finds optimal investment strategies in the class of CPPI strategies for a fund in stationarity. It is found that different investment strategies imply only modest differences in utility and, hence, that an investment collective can accommodate quite different attitudes towards risk. Finally, Kryger (2011) studies the impact of the pension design on efficiency and intergenerational fairness.

The rest of the paper is organized as follows. Section 2 presents the theoretical contributions on entrance times and moments of random walks. Section 3 describes the pension fund model, and Section 4 applies the random walk results to study bonus waiting times, the bonus size and the funding ratio. Results are given for a fund started at the bonus threshold and for a fund in stationarity. Section 5 contains a comprehensive application including a mean-variance analysis. It also explains the exact simulation algorithm used in the analysis. Finally, the appendix contains proofs for the results of Section 2 and additional lemmas.

2 Random walks

In this section we present a series of results on entrance times and conditional characteristic functions of random walks. The results will be used in subsequent sections to provide a detailed description of the distribution of bonus times, bonus size and the funding ratio of the collective pension fund model under study. However, the results are generally applicable and can be applied in many other contexts as well.

The entrance time of a random walk is defined as the (first) time of entry into $(-\infty, 0]$ after time 0. The results to follow devise how a number of quantities related to entrance times can be computed as sums over partition sets. We present both new results and existing results with new and simpler proofs. The results fall in three parts.

First, we derive a closed-form formula for the entrance time probabilities of a random walk started at the origin (Theorem 2.3). This result is also implicit in the seminal paper by Spitzer (1956), but we give here a simpler self-contained proof. Second, we derive an expression for the n^{th} derivative of the generating function for the entrance time (Theorem 2.5), which we subsequently use to derive a formula for the factorial moments (Theorem 2.6). These results are new. Third, we derive formulas for the characteristic function of the position of the random walk conditioned on entrance at time n (Theorem 2.8) and on entrance after time n (Theorem 2.9). The first of these results is known, but the proof is new, while the second result is new. Most of the proofs rely on combinatorial arguments, some of which might be of independent interest, in particular Lemma A.1.

2.1 Entrance times and partitions

Consider the random walk

$$S_0 = 0 \text{ and } S_n = S_{n-1} + X_n \text{ for } n \in \mathbb{N}, \quad (1)$$

where X_1, X_2, \dots are i.i.d. random variables. Following the notation and terminology of Asmussen (2003) we let τ_- denote the *entrance time* to $(-\infty, 0]$, also known as the first *(weak) descending ladder epoch*, defined by

$$\tau_- = \inf\{n \geq 1 : S_n \leq 0\}. \quad (2)$$

In this section our prime interest is the calculation of the entrance time probabilities

$$\tau_n = P(\tau_- = n) = P(S_1 > 0, \dots, S_{n-1} > 0, S_n \leq 0), \quad (3)$$

i.e. the probability that the entry into $(-\infty, 0]$ occurs at the n^{th} step. To facilitate the study of $(\tau_n)_{n \in \mathbb{N}}$ we introduce its generating function, defined for $0 \leq s \leq 1$ by

$$\tau(s) = \sum_{n=1}^{\infty} \tau_n s^n. \quad (4)$$

Let $p_n = P(S_n \leq 0)$ for $n \geq 1$. The following surprising theorem, originally due to Andersen (1954)¹, expresses τ in terms of the (marginal) probabilities p_n . The original proof is complicated but a simple combinatorial proof now exists, see e.g. Theorem XII.7.1 of Feller (1971).

Theorem 2.1. (Sparre Andersen Theorem) For $0 \leq s < 1$

$$\log\left(\frac{1}{1 - \tau(s)}\right) = \sum_{n=1}^{\infty} \frac{s^n}{n} p_n. \quad (5)$$

¹Erik Sparre Andersen (1919–2003) was a Danish mathematician and actuary. He played a prominent role in the design and operation of the Danish Labour Market Supplementary Pension Scheme (ATP) from its foundation in 1964 and in the years to follow. The results of this paper rest to a large extent on the work of our former colleague.

By Theorem 2.1 we can write the generating function as

$$\tau(s) = 1 - e^{H(s)}, \text{ where } H(s) = - \sum_{n=1}^{\infty} \frac{s^n}{n} p_n. \quad (6)$$

Now, since

$$\tau_n = \frac{\tau^{(n)}(0)}{n!}, \quad (7)$$

where $\tau^{(n)}$ denotes the n^{th} derivative of τ , the entrance time probabilities can in principle be calculated by repeated differentiation of expression (6). However, direct differentiation leads to an exponentially increasing number of terms (the number of terms almost triples on each iteration) so this approach is infeasible in practice for all but the smallest n . Fortunately, the number of different terms is substantially smaller. This observation gives rise to a summation formula which makes it computationally feasible to calculate τ_n for n up to at least 100. In order to state the result we define the set of integer partitions of a given order.

Definition 2.1. Define for $n \geq 1$ the partition set of order n by

$$\mathcal{D}_n = \left\{ (\sigma_1, \dots, \sigma_n) \mid \sigma_1 \in \mathbb{N}_0, \dots, \sigma_n \in \mathbb{N}_0, \sum_{i=1}^n i\sigma_i = n \right\}, \quad (8)$$

where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. For $n = 0$, \mathcal{D}_0 is the set containing the empty partition.

To illustrate the definition note that $\mathcal{D}_1 = \{(1)\}$ and $\mathcal{D}_2 = \{(2, 0), (0, 1)\}$. Integer partitions occur in number theory and combinatorics, and the size of \mathcal{D}_n as a function of n (the partition function) is a well-studied object. The following asymptotic expression is due to Hardy and Ramanujan (1918)

$$\#\mathcal{D}_n \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}. \quad (9)$$

We note that the size of \mathcal{D}_n increases sub-exponentially in n .

For later use, we define for $\sigma \in \mathcal{D}_n$ the sign of the partition and two combinatorial coefficients

$$\text{sgn}(\sigma) = (-1)^{\sum_{i=1}^n \sigma_i}, \quad d_\sigma = \prod_{i=1}^n \sigma_i! i^{\sigma_i}, \quad c_\sigma = \frac{n!}{\prod_{i=1}^n \sigma_i! (i!)^{\sigma_i}}. \quad (10)$$

We also need the following fundamental identity often used in connection with generating functions, see e.g. Chapter 7 of Szpankowski (2001) for a proof.

Theorem 2.2. Provided $\sum_{n=1}^{\infty} a_n b^n$ converges absolutely

$$\exp\left(\sum_{n=1}^{\infty} a_n b^n\right) = 1 + \sum_{m=1}^{\infty} \sum_{\sigma \in \mathcal{D}_m} \frac{b^m}{\sigma_1! \sigma_2! \dots \sigma_m!} \prod_{n=1}^m a_n^{\sigma_n}. \quad (11)$$

A combination of Theorems 2.1 and 2.2 yields a partition sum formula for the entrance time probabilities in terms of the probabilities p_n . This result can also be derived from Spitzer (1956), but the present proof is considerably simpler.

Theorem 2.3. For $n \geq 1$

$$\tau_n = - \sum_{\sigma \in \mathcal{D}_n} \frac{\text{sgn}(\sigma)}{d_\sigma} \prod_{i=1}^n p_i^{\sigma_i}. \quad (12)$$

Proof. See Appendix A.1. □

n	1	2	3	4	5	10	40	100	200
$\#\mathcal{D}_n$	1	2	3	5	7	42	37338	$1.9057 \cdot 10^8$	$3.9730 \cdot 10^{12}$
$\#\text{Naive}$	1	2	4	10	26	$5.476 \cdot 10^3$	$1.1257 \cdot 10^{18}$	$4.7720 \cdot 10^{46}$	$2.4594 \cdot 10^{94}$

Table 1: Number of terms needed to calculate the entrance time probabilities by formula (12) and (7), respectively.

Provided p_n are available Theorem 2.3 makes it feasible to calculate entrance time probabilities for fairly large values of n . Table 2.1 shows the size of \mathcal{D}_n , i.e. the number of terms in the partition sum (12), and the number of terms when (7) is used directly.² Clearly, the computational gain is massive.

Theorem 2.3 also provides the following purely combinatoric result (which is used in the proof of Lemma A.1). For $n \geq 2$

$$\sum_{\sigma \in \mathcal{D}_n} \frac{\text{sgn}(\sigma)}{d_\sigma} = 0. \quad (13)$$

This follows from (12) by considering the degenerate case $X_i \equiv 0$ in which case $p_n = 1$ for all n , $\tau_1 = 1$ and $\tau_n = 0$ for $n \geq 2$.

The coefficients d_σ obey a number of other interesting relations, e.g. the following theorem which shows that $1/d_\sigma$ can be interpreted as a probability distribution on \mathcal{D}_n . The proof of the theorem also serves as an illustration of the combinatorial method used throughout.

Theorem 2.4. *For $n \geq 1$ and $1 \leq k \leq n$*

$$\sum_{\sigma \in \mathcal{D}_n} \frac{1}{d_\sigma} = \sum_{\sigma \in \mathcal{D}_n} \frac{k\sigma_k}{d_\sigma} = 1. \quad (14)$$

2.2 Entrance time moments

In principle the entrance time moments (and other characteristics) can be calculated from the entrance time probabilities, τ_n . However, since the calculation of τ_n becomes increasingly difficult it is both theoretically and practically important to have more direct means of calculating moments. In this section we present a formula for the falling factorial entrance time moments, $E((\tau_-)_n)$, where for integers m and n we denote by $(m)_n$ the n^{th} falling factorial of m ,

$$(m)_n = m(m-1) \cdots (m-n+1). \quad (15)$$

In particular, $(m)_1 = m$ and $(m)_n = 0$ for $n > m$. Our result generalizes the well-known formula for the mean entrance time³

$$E(\tau_-) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} P(S_k > 0)\right). \quad (16)$$

The formula for the factorial moments relies on the following main result which gives a partition sum representation of the n^{th} derivative of the generating function. We denote by $f^{(n)}$ the n^{th} derivative of a function f .

Theorem 2.5. *For $n \geq 1$ and $0 \leq s < 1$*

$$\tau^{(n)}(s) = -e^{H(s)} \sum_{\sigma \in \mathcal{D}_n} c_\sigma H_\sigma(s), \quad (17)$$

where c_σ is given by (10) and

$$H_\sigma(s) = \prod_{i=1}^n \left(H^{(i)}(s)\right)^{\sigma_i}. \quad (18)$$

²The number of terms obtained by differentiating the generating function n times without collecting terms.

³Expression (16) can be derived from Theorem 2.1 by a limit argument, see e.g. Theorem XII.7.3 of Feller (1971).

Proof. See Appendix A.2. □

Note that Theorem 2.3 can be derived from Theorem 2.5 since $\tau_n = \tau^{(n)}(0)/n!$ and $H^{(i)}(0) = -(i-1)!p_i$. This constitutes an alternative proof of Theorem 2.3 which does not rely on Theorem 2.2.

By monotone convergence the n^{th} factorial moment of τ_- is given by

$$E((\tau_-)_n) = \lim_{s \rightarrow 1^-} \tau^{(n)}(s), \quad (19)$$

whether or not the limit is finite. Using Theorem 2.5 and (19) in a combination with Lemmas A.1 and A.2 the following result for the n^{th} factorial moment can be derived.

Theorem 2.6. *If $n \geq 1$ and $\sum_{k=1}^{\infty} k^{n-2}P(S_k > 0) < \infty$ then*

$$E((\tau_-)_n) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k}P(S_k > 0)\right) n! \sum_{\sigma \in \mathcal{D}_{n-1}} \prod_{i=1}^{n-1} \left(\frac{h_i}{i!}\right)^{\sigma_i} \frac{1}{\sigma_i!} < \infty, \quad (20)$$

where, for $n = 1$, the last sum is 1 by definition, and for $1 \leq i \leq n-1$

$$h_i = \sum_{k=i}^{\infty} \frac{\binom{k}{i}}{k} P(S_k > 0). \quad (21)$$

Proof. See Appendix A.2. □

By use of Theorem 2.6 we can calculate entrance time moments of any order. For the first three moments we get

$$E(\tau_-) = \exp\left(\sum_{k=1}^{\infty} \frac{q_k}{k}\right), \quad (22)$$

$$E(\tau_-^2) = E((\tau_-)_2) + E(\tau_-) = E(\tau_-) \left(2 \sum_{k=1}^{\infty} q_k + 1\right), \quad (23)$$

$$\begin{aligned} E(\tau_-^3) &= E((\tau_-)_3) + 3E((\tau_-)_2) + E(\tau_-) \\ &= E(\tau_-) \left(3 \sum_{k=2}^{\infty} (k-1)q_k + 3 \sum_{k=1}^{\infty} q_k \left[\sum_{k=1}^{\infty} q_k + 2\right] + 1\right), \end{aligned} \quad (24)$$

where $q_k = 1 - p_k = P(S_k > 0)$. These formulas are all easy to evaluate to any desired degree of accuracy.

2.3 Conditional characteristic functions

In this section we present results characterizing the position of the random walk upon entrance to $(-\infty, 0]$ (the weak descending ladder height) and the position when entrance has not yet occurred. We will need the combined generating and characteristic function defined for $|s| < 1$ and $\zeta \in \mathbb{R}$ by

$$\chi(s, \zeta) = E\left(s^{\tau_-} e^{i\zeta S_{\tau_-}}\right). \quad (25)$$

For a random variable X and an event A we write $E(X; A)$ for $E(X1_A)$, and $E(X|A)$ for $E(X1_A)/P(A)$. From Theorem VII 4.1 of Asmussen (2003) we have the following generalization of Theorem 2.1

Theorem 2.7. *For $|s| < 1$ and $\zeta \in \mathbb{R}$*

$$\log\left(\frac{1}{1 - \chi(s, \zeta)}\right) = \sum_{n=1}^{\infty} \frac{s^n}{n} E\left(e^{i\zeta S_n}; S_n \leq 0\right). \quad (26)$$

By combining Theorems 2.7 and 2.2 we obtain a partition sum formula for the characteristic function of the random walk given entrance to $(-\infty, 0]$ at time n . This result is similar to Theorem 2.3 for the entrance time probabilities.

Theorem 2.8. *For $n \geq 1$ and $\zeta \in \mathbb{R}$*

$$E(e^{i\zeta S_n} | \tau_- = n) = -\frac{1}{\tau_n} \sum_{\sigma \in \mathcal{D}_n} \frac{\text{sgn}(\sigma)}{d_\sigma} \prod_{k=1}^n (E(e^{i\zeta S_k}; S_k \leq 0))^{\sigma_k}. \quad (27)$$

Proof. See Appendix A.3. □

It is also of interest to know the distribution of the random walk given that it has not yet entered $(-\infty, 0]$. It turns out that the characteristic function for this distribution can also be calculated as a partition sum. The result is established by subtracting the characteristic function of Theorem 2.8 up to time n from the unconditional characteristic function and using Lemma A.1 to identify the resulting structure.

Theorem 2.9. *For $n \geq 1$ and $\zeta \in \mathbb{R}$*

$$E(e^{i\zeta S_n} | \tau_- > n) = \frac{1}{P(\tau_- > n)} \sum_{\sigma \in \mathcal{D}_n} \frac{1}{d_\sigma} \prod_{k=1}^n (E(e^{i\zeta S_k}; S_k > 0))^{\sigma_k}. \quad (28)$$

Proof. See Appendix A.3. □

Note that if $P(\tau_- < \infty) = 1$ then $P(\tau_- > n) = 1 - \sum_{i=1}^n \tau_i$ such that (28) can indeed be calculated.

3 Pension fund model

We consider a model for a collective pension fund with a with-profits pension product. Each contribution is split into a part giving a guaranteed payment and a part invested in a, possibly leveraged, investment portfolio. The product is with-profit in the sense that all guaranteed payments are increased, known as bonus, when the funding ratio exceeds a given threshold level. The investment strategy and the bonus policy are common and all members receive the same bonus (percentage). In our model members enter and leave the fund on financially fair terms, although this is not necessarily strictly true in practice. Despite its simplicity the model resembles the traditional collective pension funds known from e.g. the Nordic countries and the Netherlands. The random walk results presented in Section 2 will be used to give an essentially complete description of the dynamics of the fund.

First, consider a frictionless Black-Scholes market consisting of a bank account, B , with risk free short rate, r , and a risky stock, Z , with dynamics given by

$$dB(t) = rB(t)dt, \quad B(0) = 1, \quad (29)$$

$$dZ(t) = (r + \mu)Z(t)dt + \sigma_Z Z(t)dW(t), \quad Z(0) = z_0 > 0. \quad (30)$$

Here r , μ and σ_Z are strictly positive constants. The process W is a standard Brownian motion on the probability space (Ω, \mathcal{F}, P) equipped with the filtration $\mathbb{F}^W = (\mathcal{F}^W(t))_{t \geq 0}$ given by the P -augmentation of the filtration $(\sigma\{W(s); 0 \leq s \leq t\})_{t \geq 0}$.

The decision to attribute a bonus or not is taken at a set of equidistant, discrete set of time points $0 = t_0 < t_1 < \dots$. We assume for simplicity that contributions and benefits also fall at these times. The market value of the guaranteed benefits, the reserve, is denoted $R(t)$. Assuming that mortality risk can be neglected by the law of large numbers (paramount to assuming that

realized mortality equals expected mortality) the evolution of the reserve between potential bonus times is given by

$$R(t) = R(t_i)e^{r(t-t_i)}, \quad \text{where } i = \max\{j \in \mathbb{N}_0 : t_j \leq t\}. \quad (31)$$

The total assets of the fund are denoted $A(t)$ and the funding ratio of the fund is defined as

$$F(t) = \frac{A(t)}{R(t)}. \quad (32)$$

The difference between total assets and the reserve, $A(t) - R(t)$, is called the *bonus potential* (or surplus). We assume that the fund attributes bonuses according to a threshold bonus strategy such that at time t_i all guaranteed payments are increased by

$$r_i^B = \begin{cases} 0 & \text{if } F(t_i-) \leq \kappa, \\ \frac{F(t_i-) - \kappa}{\kappa} & \text{if } F(t_i-) > \kappa, \end{cases} \quad (33)$$

where κ is assumed to be strictly larger than 1. Note that immediately after a bonus attribution the funding ratio equals κ . Let \tilde{F}_i denote the funding ratio at time t_i after a (possible) bonus attribution, but before contributions and benefits have fallen. Thus $\tilde{F}_i = F(t_i-)/(1 + r_i^B) = \min\{F(t_i-), \kappa\}$.

The pension product:

- Contributions do not affect the funding ratio, i.e. for contributions received at time t_i only the fraction $1/\tilde{F}_i$ is guaranteed (at rate r) and enters the reserve while the remainder enters the bonus potential.
- The initially guaranteed benefit is entitled to bonuses from the time of contribution to the time of payment.
- Benefits do not affect the funding ratio, i.e. guaranteed benefits paid out at time t_i are increased by \tilde{F}_i (terminal bonus).

Note that the fraction of contributions guaranteed at the risk-free rate depends on the current funding ratio of the fund. Also note that for each contribution the member pays a price to enter the collective fund, but he also receives his share of the surplus for each benefit paid out.

Let c_i and b_i denote the contributions and benefits, respectively, at time t_i , and let c_i^G and b_i^G denote the part of contributions and benefits guaranteed. Total assets and the reserve at time t_i is then given by

$$A(t_i) = A(t_i-) + c_i - b_i, \quad (34)$$

$$R(t_i) = (1 + r_i^B) R(t_i-) + c_i^G - b_i^G, \quad (35)$$

where $c_i^G = c_i/\tilde{F}_i$ and $b_i^G = b_i/\tilde{F}_i$. Hence, by construction $F(t_i) = \tilde{F}_i$ irrespective of the size of contributions and benefits. It is not hard to show that the fact that pension savers enter and leave the pension fund without effecting the funding ratio makes the scheme financially fair. There is no redistribution of wealth between generations.

We assume that the pension fund has to stay funded at all times, i.e. its assets must not fall below the reserve or, equivalently, the funding ratio must not fall below one. In order to achieve this the fund pursues a CPPI (constant proportion portfolio insurance) strategy by which a constant fraction, C , of the bonus potential is invested in stocks. The remaining assets are invested in the risk-free asset. We allow for values of C greater than one, i.e. leverage of the bonus potential is possible. The dynamics of the assets between time t_i and t_{i+1} is given by

$$dA(t) = (r + \gamma(t)\mu)dt + \gamma(t)\sigma_Z dW(t), \quad (36)$$

where $\gamma(t) = C \frac{F(t)-1}{F(t)}$.

Let Δ denote the time between possible bonus attributions, $t_i = i\Delta$, and let $F_i = F(t_i)$. It follows from Preisel et al. (2010) that the funding ratio process sampled at t_i evolves like a (discrete-time) Markov chain with dynamics

$$F_i = \min \left\{ (F_{i-1} - 1) e^{(C\mu - \frac{1}{2}C^2\sigma_Z^2)\Delta + C\sigma_Z\sqrt{\Delta}U_i} + 1, \kappa \right\}, \quad (37)$$

where the U_i 's are i.i.d. standard normal variables. In particular, if $F_0 > 1$ all subsequent F_i 's are strictly larger than 1 (and at most κ).

The fund has to decide on an investment strategy, C , and a bonus policy, κ . A high bonus threshold implies that the fund can invest more freely but also that only a small fraction of the pension is guaranteed. This may or may not be in the interest of the members. Similarly, an aggressive investment strategy implies a higher probability of very high returns, but also a higher risk of very low returns (on the bonus potential).

3.1 Example

In this example we illustrate how an individual fits into the pension scheme. The pension saver enters the pension scheme (today) at age 25 and retire at age 65. Yearly pension contributions are paid until the time of retirement, with initial payment set at 2000 EUR and further contributions, assuming he is alive, increased with the risk-free rate, i.e.

$$\widehat{c}_i = 2000e^{r(i-25)}, \text{ for } i = 25, \dots, 64. \quad (38)$$

The guaranteed part of the contributions become $\widehat{c}_i^G = \widehat{c}_i/\kappa$. As mentioned, the evolution of the total reserve of the pension scheme, given by (31), assumes that realized mortality equals expected mortality. This is realistic due to the law of large numbers. However, obviously mortality cannot be neglected when considering the individual policyholder's realized cash flow. In this example we model mortality by a Gompertz-Makeham hazard rate, $u(x) = \exp((x - m)/b)/b$, with modal value and scale parameter as in Milevsky and Young (2007); $(m, b) = (88.18, 10.5)$. Let n_i denote the probability of survival until age i

$$n_i = \exp \left(- \int_{25}^i \mu(s) ds \right), \text{ for } i \geq 25.$$

By use of classic actuarial notation we can write the at age i capital value of one unit of a life annuity as

$$a_i = \sum_{j=65}^{\infty} \frac{n_j}{n_i} e^{-r(j-i)}, \text{ } i = 25, \dots, 64.$$

The part of the contribution \widehat{c}_i^G is at the time of payment turned into a guaranteed life long cash flow starting upon time of retirement. The guaranteed benefit received by the policyholder at age i , not taking bonus attributions into account and assuming he is alive, becomes (does not depend on age)

$$\widehat{b}_i^{G(\text{no bonus})} = \sum_{j=25}^{64} \frac{\widehat{c}_j^G}{a_j}, \text{ for } i = 65, \dots, \infty. \quad (39)$$

The guaranteed benefit received by the policyholder at age i , taking bonus attributions into account, and assuming he is alive, becomes

$$\widehat{b}_i^G = \sum_{j=25}^{64} \left(\frac{\widehat{c}_j^G}{a_j} \prod_{k=j+1}^i (1 + r_k^B) \right), \text{ for } i = 65, \dots, \infty. \quad (40)$$

In addition the policyholder receives a terminal bonus such that the total benefit at time i becomes $\widehat{b}_i = \kappa \widehat{b}_i^G$.

Figure 1 shows the *expected* contributions and benefits made/received by the policyholder. Relating to (34)-(35) we get $c_i = \widehat{c}_i n_i$ and $b_i = \widehat{b}_i n_i$, where c_i and b_i should be interpreted as the expected cash flows for each individual of a large cohort. We set $r = 3\%$, $\mu = 4\%$, $\kappa = 1.5$, and $C = 0$ or $C = 1.5$. We consider a simple but illustrative example where the expected return on stocks is realized every year. Thereby for C strictly positive a bonus attribution is made every year.

The expected contributions c_i , $i = 25, \dots, 64$, illustrated in Figure 1 as the positive bars, fall into two parts; the (expected) guaranteed part c_i^G (dark gray bars) and the (expected) contribution to the collective bonus potential $c_i - c_i^G$ (light gray).

The expected benefits b_i , $i = 65, \dots, \infty$, received by the policy holder, illustrated in Figure 1 as the negative bars, can be split into three parts; the (expected) upon contribution guaranteed benefits $b_i^{G(\text{no bonus})}$ (dark gray bars), (expected) benefits originating from bonus attributions $b_i^{G(\text{no bonus})} - b_i^G$, and the (expected) terminal bonus $(b_i^{G(\text{no bonus})} - b_i^G)(\kappa - 1)$.

The upper plot in Figure 1 is without investments, i.e. $C = 0$, and the lower plot corresponds to $C = 1.5$, i.e. slight leverage of the bonus potential. Although Figure 1 shows an example where the expected return on stocks is realized every year, it should be clear from Figure 1 that participating in the risky part of the investment market is expected to contribute considerable positively to the pension benefits received by the policyholder. Furthermore, note that as the retiree gets older the part of the guaranteed benefits originating from bonus attributions becomes a bigger part of the total benefit.

3.2 Collective pension funds in the real world

Real world collective pension funds are operated on a going-concern basis. The balance is composed of provisions for individual pension entitlements and collectively owned reserves (bonus potential). The latter acts both as a cushion to absorb financial and insurance risks and as a source for financing pension indexation (bonuses). Collective reserves are to be used “in the best interest of members” but typically there are no explicit objectives to guide their use. One problem faced by the fund board is that there are often conflicting interests between generations. In general, old members prefer indexation over investment freedom, while young members prefer investment freedom with the potential for long-term gains rather than immediate indexation. Thus in practice it is not obvious how to act in the best interest of (all) members in a collective scheme.

In this paper we propose to resolve the ambiguous concept of “best interest of members” by optimizing the stationary dynamics of the fund. Given the long time horizon from first contribution to last benefit for each individual member and given the indefinite time horizon of the fund itself, it can indeed be argued that “optimal” stationary dynamics should be the common objective. In the sections to follow we will characterize the stationary dynamics of the model fund and propose an optimization criterion to be evaluated in stationarity. The analysis rests on a number of simplifying assumptions and we discuss the most important of these below.

We assume that the bonus potential is only affected by investment returns and bonus attributions. In practice, however, the bonus potential must also cover other risks, in particular longevity risk. Life expectancy in the industrialized world has been increasing over the past sixty years and it continues to increase at a surprisingly fast pace, e.g. see Tuljapurkar et al. (2000). Pension funds offering guaranteed lifelong annuities are therefore exposed to substantial longevity risk. If life expectancy continues to evolve at the current pace reserves might be insufficient and the shortfall must be covered by the bonus potential. Further, regulation might require a reservation of capital to cover longevity risk which *ceteris paribus* implies less room for investment risk. Thus in practice longevity risk is likely to influence the investment profile of the fund. On the other hand, there are also pension funds offering only lump sum payments at retirement or where benefits are linked to the life expectancy experience. These funds are close

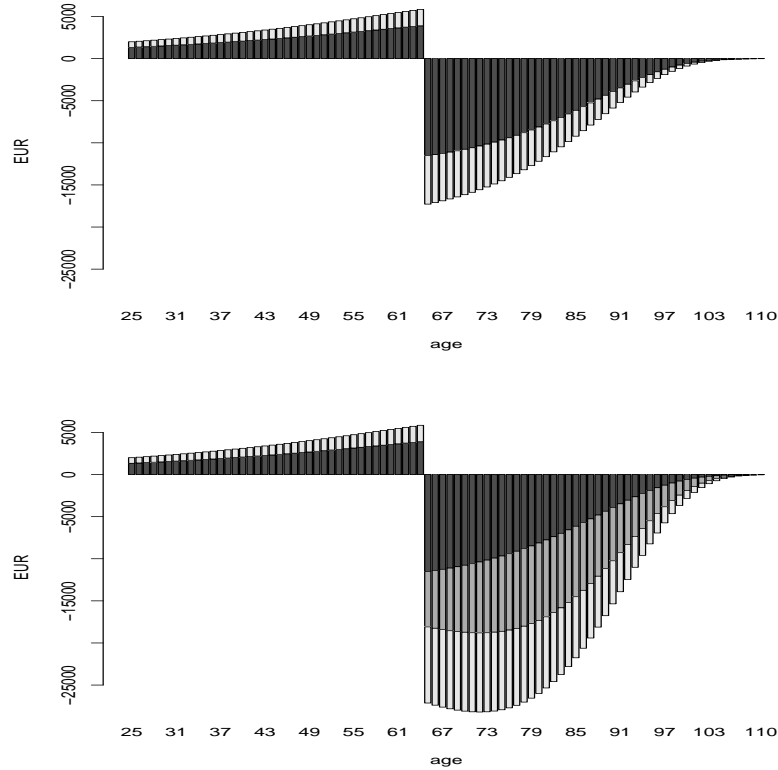


Figure 1: The expected cash flow of an individual policy holder: Positive parts, from age 25 to 64, are expected pension contributions, and negative parts, from age 65 to 110, are expected benefits. Dark gray bars: Original guaranteed parts. Gray bars: Increased guaranteed benefits due to bonuses. Light gray bars: Bonus potential contributions and terminal bonus benefits, respectively. Upper plot corresponds to $C = 0$, and lower plot corresponds to $C = 1.5$.

to fulfilling our assumption of no longevity risk.

Longevity risk is a systematic risk that affects an entire member base regardless of its size. In addition to this risk, a real world pension fund also faces so-called unsystematic mortality risk caused by the random nature of death at the individual level. However, in contrast to longevity risk the unsystematic risk is diversifiable across members and will cause only minor fluctuations in the benefit outflow for a large fund. We implicitly make the assumption that the pension fund under consideration is so large that unsystematic mortality risk can safely be ignored.

Another key assumption is the ability of the fund to perfectly hedge its liabilities. In the Black-Scholes market there is a single, constant interest rate, and all nominal payments can therefore be replicated (hedged) by a cash deposit in the bank account. The constant rate implies that the reserve amortizes at the same rate irrespective of when the benefits fall due, and consequently we do not need to take the profile of the underlying benefit cash flow into account. Realism could be added by introducing a stochastic interest rate model. This would give rise to a term structure and liabilities should then be hedged by a portfolio of zero-coupon bonds. The interest rate sensitivity of the reserve complicates the analysis, but as long as the liability can be hedged the situation is essentially the same as the one considered.

In practice, however, the reserve is typically *not* calculated on a tradable market curve, and the reserve therefore does not represent the value of a financial hedge. For example, the discount curve under the forthcoming Solvency II regulatory framework in Europe is an intricate construction which is only partly market based, and there is no guarantee that a “Solvency II

reserve” is sufficient to hedge the liability. Simply put, our analysis is concerned with the financial value of liabilities, not the reported value.

In line with the vast majority of the related literature, we assume that the fund can operate without frictions in the capital markets. This implies that the fund can lever the bonus potential and also derisk fast enough to remain fully funded in all situations. Leverage through the use of highly liquid index futures are both available and used in real life. However, actual markets do not evolve continuously and there is a risk of the fund incurring larger losses in reality than in the model. To fully control the downside risk, the fund can use options either as tail protection or to obtain one-sided stock market participation, but both of these strategies come at the price of lower expected returns. Many pension funds also invest in illiquid assets which reduce investment flexibility. It seems fair to say that most pension funds do not adapt their risk exposure as dynamically as assumed in the analysis, but it is more of a choice than a necessity.

Finally, we make the assumption that the funding ratio is invariant to contributions entering and benefits leaving the fund. This is a strict version of the reasonable requirement that there should be no systematic redistribution of wealth between generations. Of course, in practice, this requirement is interpreted somewhat more loosely. From a mathematical point of view the “funding invariance” is important, because it allows us to ignore the underlying demographics. Otherwise, we would have to explicitly model the contribution profile over an entire population and its evolution over time. The assumption on contributions, benefits and bonus attributions all taken place at the same time is however made only for notational convenience and can easily be relaxed.

3.3 Outline

The purpose of the rest of the paper is twofold. First, we characterize the impact of C and κ in terms of bonus time (time between bonus attributions), bonus size and funding ratio. The characterization is provided for funds started at κ and in stationarity (the long-run average).

Second, we propose a criterion by which κ and C can be determined. The criterion is evaluated in stationarity to reflect the fact that the fund is collective and should be designed for the benefit of the average member. The simple pension product to be considered is the case in which a contribution of one is made at time $t = 0$ for a benefit paid out in its entirety at $t = T$ (retirement). In this case the benefit at retirement becomes

$$O_T = \frac{F(T)}{F(0)} e^{rT} \prod_{i=1}^T (1 + r_i^B). \quad (41)$$

In principle, different contribution and benefit profiles, e.g. as in Subsection 3.1, could also be analyzed. The point to note is that the payoff depends solely on the funding ratio dynamics.

4 Bonus and funding ratio

The time between consecutive bonuses, the size of the bonus and the funding ratio given no bonus has yet been attributed can all be analyzed by the random walk results of Section 2.

Consider the following transformation

$$Y_n = -\log\left(\frac{F_n - 1}{\kappa - 1}\right) \text{ for } n \in \mathbb{N}_0. \quad (42)$$

This transformation turns the funding ratio process (37) into the one-sided random walk

$$Y_n = (Y_{n-1} + X_n)^+, \quad (43)$$

where the X_n 's are i.i.d. normally distributed with mean $-(C\mu - \frac{1}{2}C^2\sigma_Z^2)\Delta$ and variance $C^2\sigma_Z^2\Delta$. Note that $F_n = \kappa$ corresponds to $Y_n = 0$, while funding ratios close to one correspond to high values of Y .

Along with Y we also consider the (unrestricted) random walk of Section 2,

$$S_0 = 0 \text{ and } S_n = S_{n-1} + X_n \text{ for } n \in \mathbb{N}, \quad (44)$$

with the same X_n 's as in (43). The distribution of S_n is given by

$$S_n \sim N\left(-n\left(C\mu - \frac{1}{2}C^2\sigma_Z^2\right)\Delta, nC^2\sigma_Z^2\Delta\right). \quad (45)$$

Thus, in the notation of Section 2 we have

$$p_n = P(S_n \leq 0) = \Phi\left(\frac{\sqrt{n}\Delta\left(\mu - \frac{1}{2}C\sigma_Z^2\right)}{\sigma_Z}\right), \quad (46)$$

where Φ denotes the cumulative distribution function (CDF) of a standard normal distribution.

4.1 Stationarity

The first question of interest is whether the fund admits a stationary distribution or not. In the stationary case the funding ratio distribution converges towards a non-degenerate distribution, otherwise it converges (in probability) towards one. The following result answers the question in terms of the aggressiveness of the investment strategy (the result was also by Preisel et al. (2010) albeit in a different parametrization).

Proposition 4.1. *The funding ratio process (37) admits a stationary distribution if and only if $C < \frac{2\mu}{\sigma_Z^2}$.*

Proof. By Proposition 11.5.3 of Meyn and Tweedie (2009) we have that the Y -process, and hence the F -process, admits a stationary distribution iff the mean of the increments X_n is strictly negative, i.e. iff $C < \frac{2\mu}{\sigma_Z^2}$. \square

When it exists, we will denote the stationary funding ratio distribution by π . We note that the existence of a stationary distribution is independent of how often (Δ) and at which level (κ) a bonus is allotted. A stationary distribution exists if and only if the median return on the bonus potential is positive. If the bonus potential is invested more aggressively than that it will eventually get lost (in the boundary case, $C = 2\mu/\sigma_Z^2$, bonus will in fact be attributed infinitely often, but the average time between each bonus is infinite!).

4.2 Bonus times

We refer to the time between consecutive bonus attributions as bonus times, or more precisely bonus waiting time. Formally, these are defined by

$$\mathcal{T}_1 = \inf\{n \geq 1 : F_n = \kappa\} = \inf\{n \geq 1 : Y_n = 0\}, \quad (47)$$

and, recursively, for $k \geq 2$

$$\mathcal{T}_k = \inf\{n \geq 1 : F_{\mathcal{T}_{k-1}+n} = \kappa\} = \inf\{n \geq 1 : Y_{\mathcal{T}_{k-1}+n} = 0\}. \quad (48)$$

Consider first the case where $F_0 = \kappa$. Then $Y_0 = 0$ and the first bonus time coincides with the entrance time of S to $(-\infty, 0]$, i.e. $\mathcal{T}_1 = \tau_-$, where τ_- is given by (2) of Section 2. Further, since the funding ratio is κ after a bonus attribution it follows by the Markov property that all subsequent bonus times are independent and distributed as \mathcal{T}_1 .

Consider next the stationary case and assume that F_0 is distributed according to the stationary funding ratio distribution. Imagine that the fund has been operating since time minus infinity. The probability that at time 0 we are in a period (between two bonuses) of length k is then given by $k\tau_k / \sum_{n=1}^{\infty} n\tau_n$, i.e. the probability is proportional to the length of the period times the frequency by which it occurs. Further, given that we are in a period of length k the probability that we are n places ($n \leq k$) away from the end is $1/k$, since each position is equally likely. Summing over all possible k 's we get that the probability that the next bonus occurs at time $n \geq 1$ is given by

$$P_{\pi}(\mathcal{T}_1 = n) = \sum_{k=n}^{\infty} \frac{1}{k} \left(\frac{k\tau_k}{\sum_{n=1}^{\infty} n\tau_n} \right) = \frac{\sum_{k=n}^{\infty} \tau_k}{E(\tau_-)} = \frac{1 - \sum_{k=1}^{n-1} \tau_k}{E(\tau_-)}, \quad (49)$$

where we use subscript π to denote that F_0 is drawn from the stationary distribution. When the first bonus (after time 0) has been attributed the funding ratio is κ . Hence, all subsequent bonus times are distributed as τ_- .

Using subscript κ to denote the case $F_0 = \kappa$ we thus have

Proposition 4.2. *For $k \geq 1$ and $n \geq 1$*

$$P_{\kappa}(\mathcal{T}_k = n) = \tau_n, \quad (50)$$

and, provided the stationary distribution exists,

$$P_{\pi}(\mathcal{T}_k = n) = \begin{cases} \left(1 - \sum_{k=1}^{n-1} \tau_k\right) / E(\tau_-) & \text{for } k = 1, \\ \tau_n & \text{for } k \geq 2, \end{cases} \quad (51)$$

where τ_n is given by (12) of Theorem 2.3 and $E(\tau_-)$ is given by (16) of Section 2.2

Note that in stationarity the probability of receiving a bonus in any given year is

$$P_{\pi}(r_1^B > 0) = P_{\pi}(Y_1 = 0) = P_{\pi}(\mathcal{T}_1 = 1) = \frac{1}{E(\tau_-)}. \quad (52)$$

This relationship is also known as Kac's theorem.

In the stationary case the drift of S is negative and it is not hard to show that the criterion of Theorem 2.6 is satisfied for all n . Hence, the time between bonuses has factorial moments of all orders and these can be calculated by (20) of Theorem 2.6.⁴

4.3 Number of bonuses

The number of bonuses in a given period can be calculated from the bonus time distribution. Let $R_k = \mathcal{T}_1 + \dots + \mathcal{T}_k$ denote the time of the k 'th bonus, also known as the renewal epochs. For $F_0 = \kappa$ and $F_0 \sim \pi$ the distribution of R_k can be calculated by the recursion

$$P_*(R_1 = n) = P_*(\mathcal{T}_1 = n), \quad (53)$$

$$P_*(R_k = n) = \sum_{j=k-1}^{n-1} P_*(R_{k-1} = j)\tau_{n-j}. \quad (54)$$

where $P_*(\mathcal{T}_1 = n)$ is given by Proposition 4.2 (and $*$ is either κ or π).

Let N_n denote the number of bonuses from time 1 to time n . We have

$$P_*(N_n \leq k) = P_*(R_{k+1} > n) = 1 - \sum_{j=k+1}^n P_*(R_{k+1} = j), \quad (55)$$

where $*$ is either κ or π .

⁴In fact, it can be shown that Y is so-called geometrically ergodic which implies exponential moments of the return time to 0, i.e. the time between bonuses.

4.4 Bonus percentage

The bonus percentage distribution can be derived from the (descending) ladder height distribution of the random walk. For $F_0 = \kappa$ the events $(\mathcal{T}_1 = n)$ and $(\tau_- = n)$ are identical and on this event the bonus percentage given by (33) becomes

$$r_n^B = \frac{\kappa - 1}{\kappa} (e^{-S_n} - 1). \quad (56)$$

The mean bonus percentage given the time of bonus can then be calculated by use of Theorem 2.8.

Proposition 4.3. *For $n \geq 1$*

$$E_\kappa (r_n^B | \mathcal{T}_1 = n) = \frac{\kappa - 1}{\kappa} [E(e^{-S_n} | \tau_- = n) - 1], \quad (57)$$

where

$$E(e^{-S_n} | \tau_- = n) = -\frac{1}{\tau_n} \sum_{\sigma \in \mathcal{D}_n} \frac{\text{sgn}(\sigma)}{d_\sigma} \prod_{k=1}^n (E(e^{-S_k}; S_k \leq 0))^{\sigma_k} \quad (58)$$

with

$$E(e^{-S_k}; S_k \leq 0) = e^{kC\mu\Delta} \Phi\left(\sqrt{k\Delta} \frac{\mu + \frac{1}{2}C\sigma_Z^2}{\sigma_Z}\right). \quad (59)$$

Further, for $n \geq 1$ and provided the stationary distribution exists

$$E_\pi (r_n^B | \mathcal{T}_1 = n) = \frac{1}{P_\pi(\mathcal{T}_1 = n)} \sum_{k=0}^{\infty} E_\kappa (r_{n+k}^B | \mathcal{T}_1 = n+k) \frac{\tau_{k+n}}{E(\tau_-)}. \quad (60)$$

Proof. Formula (57) follows directly from (56). Theorem 2.8 identifies the conditional distribution of S_n given $\tau_- = n$ as a linear combination of conditional normal tail measures. Since the normal distribution has exponential moments of all orders we conclude (58) by dominated convergence. Expression (59) follows from (45) by standard calculations.

For the stationary case we consider a fund which has been run since time minus infinity. Let λ denote time since a bonus was last attributed,

$$\lambda = \inf\{n \geq 0 : F_{-n} = \kappa\} = \inf\{n \geq 0 : Y_{-n} = 0\}. \quad (61)$$

By the argument of Section 4.2 leading to (49) and the Markov property we have

$$P_\pi(\mathcal{T}_1 = n, \lambda = k) = \frac{1}{n+k} \frac{(n+k)\tau_{k+n}}{E(\tau_-)} = \frac{\tau_{k+n}}{E(\tau_-)}, \quad (62)$$

$$E_\pi(r_n^B | \mathcal{T}_1 = n, \lambda = k) = E_\kappa(r_{n+k}^B | \mathcal{T}_1 = n+k). \quad (63)$$

By summing over all possible values of λ we obtain

$$\begin{aligned} E_\pi(r_n^B | \mathcal{T}_1 = n) &= \frac{1}{P_\pi(\mathcal{T}_1 = n)} E_\pi(r_n^B; \mathcal{T}_1 = n) \\ &= \frac{1}{P_\pi(\mathcal{T}_1 = n)} \sum_{k=0}^{\infty} E_\pi(r_n^B; \mathcal{T}_1 = n, \lambda = k) \\ &= \frac{1}{P_\pi(\mathcal{T}_1 = n)} \sum_{k=0}^{\infty} E_\pi(r_n^B | \mathcal{T}_1 = n, \lambda = k) P_\pi(\mathcal{T}_1 = n, \lambda = k). \end{aligned} \quad (64)$$

Finally, inserting (62) and (63) in (64) yields (60). \square

Higher order polynomial moments of r^B can be expressed in terms of exponential moments of S_n by expanding (56). It is straightforward to extend Proposition 4.3 to cover this case also.

4.5 Funding ratio

We consider at last the funding ratio of the fund given that no bonus has yet been attributed. For $F_0 = \kappa$ the events $(\mathcal{T}_1 > n)$ and $(\tau_- > n)$ are identical and on this event

$$F_n = (\kappa - 1)e^{-S_n} + 1. \quad (65)$$

Polynomial funding ratio moments can be derived by use of Theorem 2.9. For the mean funding ratio we have the following result.

Proposition 4.4. *For $n \geq 1$*

$$E_\kappa (F_n | \mathcal{T}_1 > n) = (\kappa - 1)E(e^{-S_n} | \tau_- > n) + 1, \quad (66)$$

where

$$E(e^{-S_n} | \tau_- > n) = \frac{1}{P(\tau_- > n)} \sum_{\sigma \in \mathcal{D}_n} \frac{1}{d_\sigma} \prod_{k=1}^n (E(e^{-S_k}; S_k > 0))^{\sigma_k} \quad (67)$$

with

$$E(e^{-S_k}; S_k > 0) = e^{kC\mu\Delta} \Phi\left(-\sqrt{k\Delta} \frac{\mu + \frac{1}{2}C\sigma_Z^2}{\sigma_Z}\right). \quad (68)$$

Further, for $n \geq 1$ and provided the stationary distribution exists

$$E_\pi (F_n | \mathcal{T}_1 > n) = \frac{1}{P_\pi(\mathcal{T}_1 > n)} \sum_{k=0}^{\infty} E_\kappa (F_{n+k} | \mathcal{T}_1 > n+k) P_\pi(\mathcal{T}_1 = n+k+1), \quad (69)$$

Proof. We will only prove (69). With λ as in (61) and by use of (62) we have

$$P_\pi(\mathcal{T}_1 > n, \lambda = k) = \sum_{i=n+1}^{\infty} P_\pi(\mathcal{T}_1 = i, \lambda = k) = \sum_{i=n+1}^{\infty} \frac{\tau_{i+k}}{E(\tau_-)} = P_\pi(\mathcal{T}_1 = n+k+1), \quad (70)$$

and, by the Markov property,

$$E_\pi (F_n | \mathcal{T}_1 > n, \lambda = k) = E_\kappa (F_{n+k} | \mathcal{T}_1 > n+k). \quad (71)$$

Then

$$\begin{aligned} E_\pi (F_n | \mathcal{T}_1 > n) &= \frac{1}{P_\pi(\mathcal{T}_1 > n)} E_\pi (F_n; \mathcal{T}_1 > n) \\ &= \frac{1}{P_\pi(\mathcal{T}_1 > n)} \sum_{k=0}^{\infty} E_\pi (F_n; \mathcal{T}_1 > n, \lambda = k) \\ &= \frac{1}{P_\pi(\mathcal{T}_1 > n)} \sum_{k=0}^{\infty} E_\pi (F_n | \mathcal{T}_1 > n, \lambda = k) P_\pi(\mathcal{T}_1 > n, \lambda = k). \end{aligned} \quad (72)$$

By inserting (70) and (71) into (72) we obtain (69). \square

5 Applications

In the following we calculate key statistics for the pension fund model of Section 3, and we illustrate how these statistics are influenced by the bonus threshold (κ) and the investment strategy (C). Based on the results of Section 4 we calculate the bonus time distribution, the bonus time moments, the number of bonuses, the expected bonus and the expected funding ratio given no bonus. Statistics are calculated for a fund at the bonus threshold and for a fund in stationarity.

In Section 5.2 we consider a pension saver paying one monetary unit to the pension fund and receiving 40 years later his terminal pension benefit as a lump sum. Closed form expressions for the pension benefit mean and variance are derived for a fund at the bonus threshold at the time of the contribution. Further, we present in Proposition 5.1 an exact simulation algorithm which allows the calculation of the pension benefit mean and variance in stationarity. This is used to find the investment strategy optimizing the expected payout in stationarity for given bonus threshold, i.e. the expected payout for the average saver.

Exact samples from the stationary distribution can also be obtained by the algorithm of Ensor and Glynn (2000). Their algorithm uses exponential tilting and requires exponential moments of the innovation distribution to generate independent, identically distributed samples. In contrast, our algorithm generates partly dependent, identically distributed samples with no distributional assumptions.

We assume throughout that a bonus is (possibly) attributed once a year ($\Delta = 1$), and we use the following capital market parameters $r = 3\%$, $\mu = 4\%$, and $\sigma_Z = 15\%$.

5.1 Characterization

As a base case example we choose $C = 1.5$ and $\kappa = 1.5$. Thus when the fund is at the bonus threshold $2/3$ of contributions are guaranteed the risk-free rate r . For contributions committed to the fund at lower funding ratios, i.e. in periods between bonuses, a larger fraction is guaranteed.

Note that since C is larger than one the bonus potential is leveraged, i.e. the amount invested in stocks is larger than the bonus potential. At the bonus threshold the fraction of total assets invested in stocks is given by $(1 - (1/\kappa))C$. For a base case fund at the bonus threshold a fraction of $1/2$ of total assets are invested in stocks.

Stationarity

Only investment strategies which give rise to a stationary funding ratio process are considered viable options for a collective pension fund. Otherwise the funding ratio will (essentially) converge to one implying that all assets are invested in the risk-free asset only or, equivalently, that all contributions are fully guaranteed. Since one of the purposes of entering a collective fund is to get access to the capital market in a cost-effective way, the latter situation defies the purpose of an investment collective.⁵

Proposition 4.1 provides an upper bound on the investment strategy, C , for a stationary distribution to exist. The bound depends on the capital market parameters only, and neither on the threshold (κ) nor the frequency of possible bonus attributions (Δ). For the capital market parameters stated above the fund admits a stationary distribution if and only if C is at most 3.56. Hence, with $C = 1.5$ the base case fund is stationary.

For higher values of σ_Z and/or smaller values of μ the upper bound is appreciably smaller. For the higher, but not unrealistic, volatility of $\sigma_Z = 20\%$ and with the same risk-premium of $\mu = 4\%$ the upper bound on C is 2. For a base case fund at the threshold the bonus potential constitutes one third of total assets. In this case, a bound of 2 implies that the fund can invest at most two thirds of its assets in stocks. Thus, the stationarity requirement can impose material constraints on the investment strategy. Figure 2 illustrates the upper bound on C for different market parameter sets.

Bonus times

The first bonus time, \mathcal{T}_1 , measures the time of the first bonus after time zero. For a fund initially at the bonus threshold the distribution of \mathcal{T}_1 can be identified as the distribution of τ_- , i.e. the entrance time to $(-\infty, 0]$ of the associated random walk. For a fund in stationarity, however, it

⁵The other main purpose of a collective fund is the ability to provide lifelong benefit streams through “diversification” of the individual member’s time of death. However, in the current paper we do not consider this aspect.

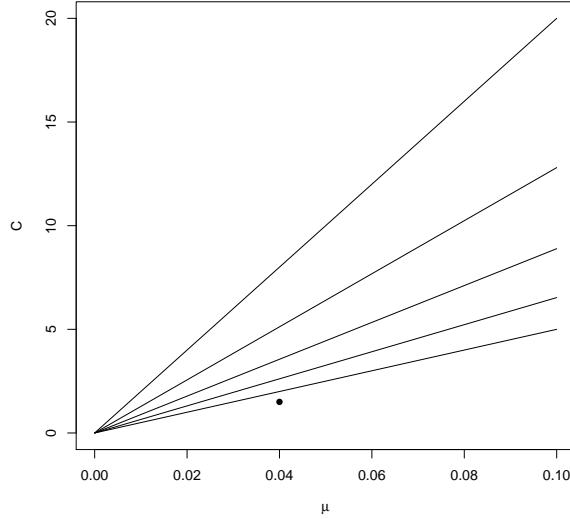


Figure 2: Upper bound on the fraction of the bonus potential invested in risky assets (C) for a stationary funding ratio process to exist. The bound is shown as a function of expected excess return of the risky asset (μ) for selected values of volatility (σ_Z). From highest to lowest the bounds correspond to $\sigma_Z = 10\%$, 12.5% , 15% , 17.5% and 20% . The dot indicates the base case values of $(\mu, C) = (4\%, 1.5)$.

typically takes longer before the first bonus is attributed since the funding ratio at time zero is often below the bonus threshold. Once the first bonus is attributed, the waiting time between all subsequent bonuses is distributed as τ_- regardless of the funding ratio at time zero.

It turns out, perhaps somewhat surprising, that for a fund starting out either at the bonus threshold or in stationarity the distribution of \mathcal{T}_1 depends only on the investment strategy, and not on the bonus threshold.⁶ The top left plot of Figure 3 shows the distribution of \mathcal{T}_1 in these two cases as given by Proposition 4.2. For a base case fund at the bonus threshold the probability of a bonus at the first year is over fifty percent, while the same probability in stationarity is only about twenty percent. The large difference between these values implies that in stationarity the fund is typically “between bonuses”.

The probability of a bonus at the first year as a function of the investment strategy is shown in the bottom left plot of Figure 3. For a fund at the bonus threshold, the probability is over fifty percent for all considered investment strategies albeit decreasing in C . The stationary probability on the other hand tends to zero as C approaches the upper bound for stationarity of 3.56. Recall that the stationarity probability also has the interpretation as the long-term average, i.e. the frequency with which bonuses will be attributed over long horizons (regardless of the initial funding status).

Moments of the time between bonuses can be calculated by Theorem 2.6. The mean and standard deviation for various values of C are shown in Table 2; as mentioned above the distribution and hence the moments depend only on the investment strategy. In the base case the mean is five years, but with a standard deviation of almost fourteen years. Thus there is considerable variability in the length of the periods between bonuses. For larger values of C the mean and, in particular, the standard deviation increase. A fund with $C = 3$ is still stationary but there will be decades, and even centuries, with no bonus. However, by Proposition 4.1 we have that

⁶However, for a fund starting out at a given funding ratio (below the bonus threshold) the distribution of \mathcal{T}_1 does of course depend on the bonus threshold also.

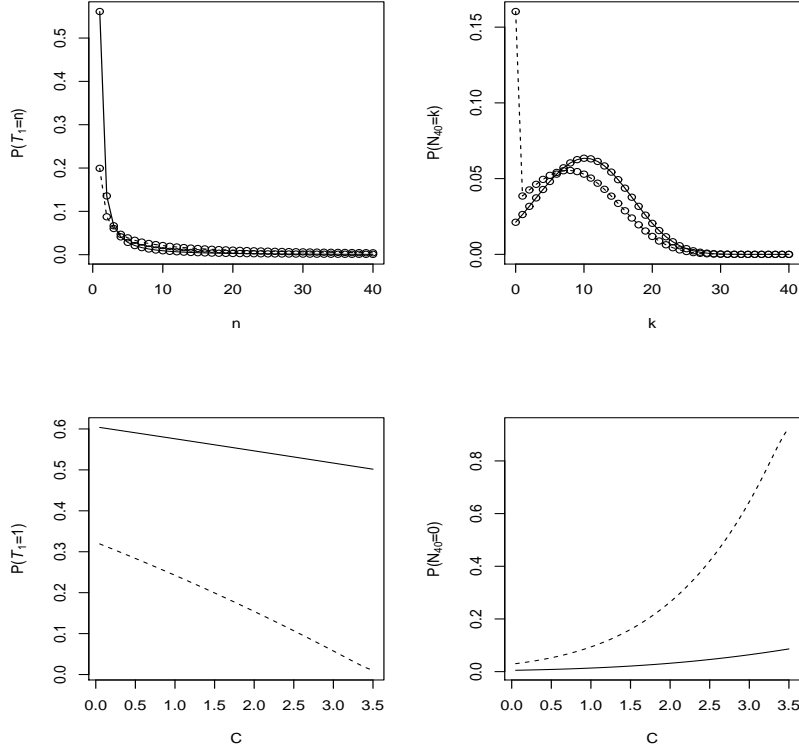


Figure 3: Solid curves correspond to $F_0 = \kappa$ and dashed curves to the stationary case, $F_0 \sim \pi$. For the upper plots $C = 1.5$. Upper left: Distribution of the first bonus. Lower left: Probability of a bonus at the first year as a function of the investment strategy C . Upper right: Distribution of the number of bonuses in 40 years. Lower right: Probability of no bonuses in 40 years as a function of the investment strategy C .

the funding ratio admits a stationary distribution iff the median return on the bonus potential is positive (for a base case fund that is $C < 3.56$). In other words, the stationarity requirement imposes a median time between bonuses⁷ of 1. We conclude that the distribution of the waiting times between bonuses is heavily right skewed.

C	1	1.5	2	2.5	3
$E(\tau_-)$	4.12	5.02	6.49	9.35	17.39
$SD(\tau_-)$	9.87	13.73	20.93	37.55	98.60

Table 2: The mean and standard deviation of the time between bonuses for different investment strategies C . In all cases the median is 1.

Number of bonuses

The number of bonuses from time 1 to time n is denoted N_n . The distribution of N_n can be calculated from the distributions of \mathcal{T}_1 and τ_- as described in Section 4.3. For a fund starting out either at the bonus threshold or in stationary the distribution of N_n depends only on the investment strategy, and not on the bonus threshold.

⁷ Defined as $\min\{n \in \mathbb{N} : \sum_{i=1}^n \tau_i \geq 0.5\}$

The top right plot of Figure 3 shows the distribution of N_{40} for a base case fund. For a base case fund starting out at the bonus threshold the distribution of N_{40} is unimodal and the number of bonuses will most likely be around 10. In the stationarity case, however, there is an additional peak at zero. There is thus a rather large probability of about fifteen percent of no bonus at all in forty years, corresponding to the events where the fund is initially at a (very) low funding level. If a bonus is attributed at least once the fund evolves like a fund started at the bonus threshold for the remaining period. The part of the stationary distribution of N_{40} at one and above therefore looks like a scaled and shifted version of the “threshold” distribution.

The probability of no bonuses in forty years as a function of the investment strategy is depicted in the bottom right plot of Figure 3. For increasing C , the probability is modestly increasing for a fund at the bonus threshold, while the stationary probability increases to one.

Bonus percentage

In contrast to the time of bonus, the bonus percentage (conditioned on a bonus being given) depends on both the investment strategy and the bonus threshold. It also depends on the time since the last bonus.

The bonus percentage is determined by the funding ratio distribution of the year prior; the higher the funding ratio the higher the conditional expected bonus.⁸ If a fund starting at the bonus threshold does not give a bonus in the first year the funding ratio will be strictly below the bonus threshold. This implies that a bonus in the second year (if given) is on average smaller than a bonus in the first year (if given). This argument is hard to continue formally, but it seems at least intuitively reasonable that the expected bonus will be decreasing in the time since the last bonus.

The bonus percentage conditioned on the value of \mathcal{T}_1 can be calculated by Proposition 4.3. It is shown in the top left plot of Figure 4 for the base case fund. We see that the conditional expected bonus quickly drops by one to two percentage points depending on how the fund is started, and then levels off to just below 5.5%.

The middle left plot of Figure 4 shows the expected value of the first bonus,

$$E(r_{\mathcal{T}_1}^B) = \sum_{n=1}^{\infty} E(r_n^B | \mathcal{T}_1 = n) P(\mathcal{T}_1 = n), \quad (73)$$

for the base case fund. The expected bonus (when given) is increasing in C , both in stationarity and for a fund starting at the bonus threshold. In stationarity, however, the frequency with which bonuses are attributed decreases with C , cf. lower left plot of Figure 3. The average bonus in stationarity is therefore a trade-off between many, small bonuses and few, large bonuses. The average bonus in stationarity,

$$E_{\pi}(r^B) = E_{\pi}(r_1^B | \mathcal{T}_1 = 1) P_{\pi}(\mathcal{T}_1 = 1), \quad (74)$$

is shown in the lower left plot of Figure 4. It is seen that the long-term average bonus is maximized for C just below 2.

It follows from Proposition 4.3 that the average bonus for funds with the same C but different bonus thresholds are linearly related. Specifically, the average bonus in stationarity is related by

$$E_{\pi}^{(C, \kappa_2)}(r^B) = \frac{\kappa_2 - 1}{\kappa_2} \frac{\kappa_1}{\kappa_1 - 1} E_{\pi}^{(C, \kappa_1)}(r^B). \quad (75)$$

This implies that plots for different thresholds are scaled versions of each other. In particular, the average bonus in stationarity is maximized for the same C . It also follows that the average bonus for a fund with $\kappa = 3$ is twice as high as the average bonus for a fund with $\kappa = 1.5$ (base case value). Of course, the guaranteed part to which the bonus is applied is correspondingly smaller.

⁸This in fact is not obvious, since we condition on a bonus being given. It can nevertheless be shown as a consequence of stochastic ordering and log-concavity of the normal distribution.

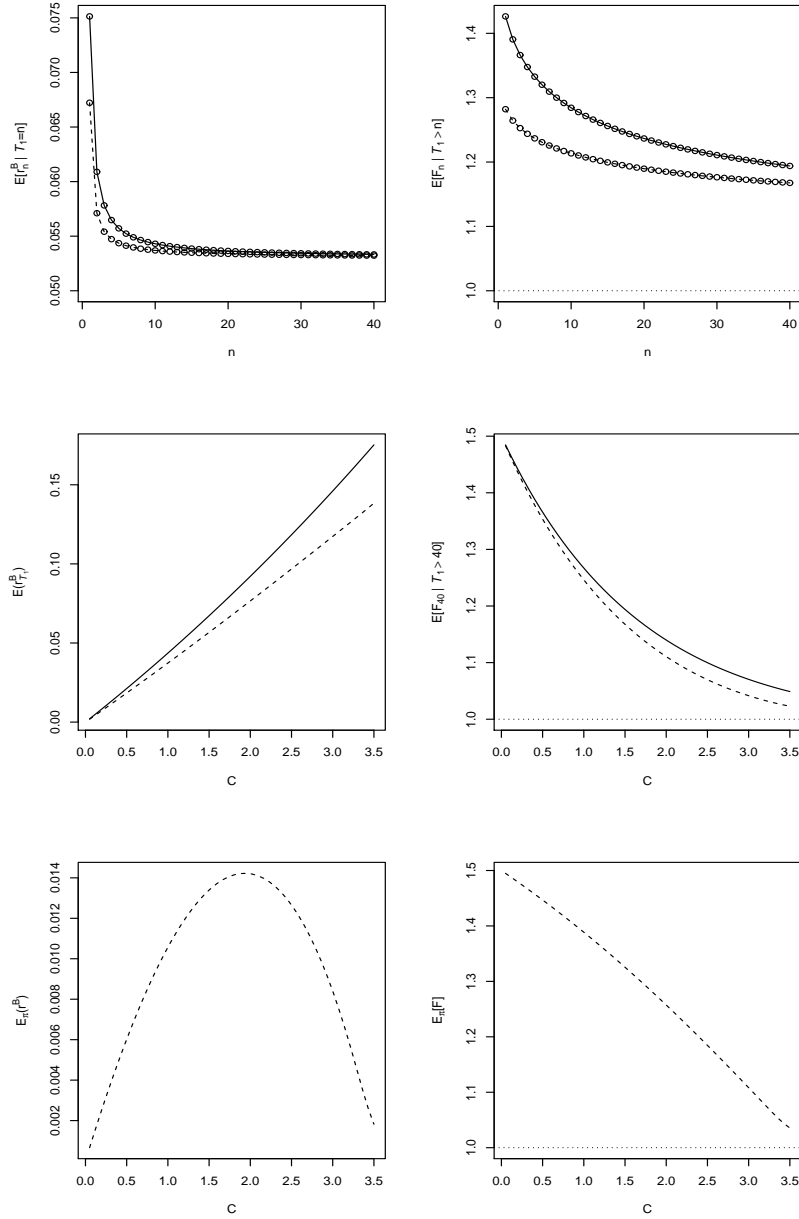


Figure 4: Solid curves correspond to $F_0 = \kappa$ and dashed curves to the stationary case, $F_0 \sim \pi$. Bonus threshold $\kappa = 1.5$ in all plots. Upper left: Conditional expected bonus percentage ($C = 1.5$). Middle left: Expected first bonus. Lower left: Expected bonus in stationarity. Upper right: Conditional expected funding ratio given no bonus ($C = 1.5$). Middle right: Conditional expected funding ratio given no bonus in 40 years. Lower right: Expected funding ratio in stationarity.

Funding ratio

The pension payoff depends on the funding ratio when contributions are committed, the bonuses up to the time of payout, and the funding ratio at payout. Thus to evaluate the payoff we need to consider both the funding ratio at the time money enters the fund, and the funding ratio at

the time money leaves the fund.

The expected funding ratio as a function of the time since the last bonus can be calculated by Proposition 4.4. This is shown for the base case fund in the upper right plot of Figure 4. The expected funding ratio is decreasing in the time since the last bonus. Intuitively, this seems reasonable since absence of a bonus indicates that the fund is experiencing poor investment results. It is perhaps surprising, however, that the (expected) funding ratio seems to level off, at around 120%. Thus beyond a certain point the funding ratio does not deteriorate any further. Limiting distributions of Markov chains conditioned on non-absorption (or in our case no bonus) are known as Yaglom limits. We conjecture that the fund possesses a Yaglom limit both in stationarity and when started at the threshold, i.e. that the funding ratio distribution conditioned on no bonus converges to a non-degenerate distribution. However, establishing existence, let alone identifying, Yaglom limits is non-trivial and a formal study is outside the scope of this paper. The interested reader is referred to the specialist literature on quasi-stationarity, e.g. Tweedie (1974); Jacka and Roberts (1995); Lasserre and Pearce (2001).⁹

We know that high equity exposures lead to high, but infrequent, bonuses. It also leads to low expected funding ratios. The expected funding ratio in stationarity,

$$E_\pi(F) = \kappa P_\pi(\mathcal{T}_1 = 1) + E(F_1 | \mathcal{T}_1 = 1) P_\pi(\mathcal{T}_1 > 1), \quad (76)$$

is shown in the lower right plot of Figure 4, while the expected funding ratio after 40 years with no bonus is shown in the middle right plot. For low values of C both the unconditional and conditional expected funding ratios are close to the maximum of 1.5, while as C approaches the upper limit for stationarity the (expected) funding ratio tends to one.

We finally note that it follows from Proposition 4.4 that the expected funding ratio for funds with the same C but different thresholds are related by an affine transformation. Specifically, the expected funding ratio in stationarity is related by

$$E_\pi^{(C, \kappa_2)}(F) = \frac{\kappa_2 - 1}{\kappa_1 - 1} \left(E_\pi^{(C, \kappa_1)}(F) - 1 \right) + 1. \quad (77)$$

Higher bonus thresholds thereby lead to higher expected funding ratios and hence lower guarantees.

5.2 Pension benefits

The rationale behind guaranteeing a fixed return on a part of the contributions is that it ensures a certain minimum benefit. However, guarantees reduce the risk capacity for risky assets impairing expected returns. Expected returns can be increased by leverage of the bonus potential, but this in turn increases the variability. The (minimum) fraction guaranteed and the expected return/variability are controlled by κ and C , respectively.

In this section we calculate the mean and variance of the payout considered in Section 3 (repeated here for ease of reference),

$$O_T = \frac{F_T}{F_0} e^{rT} \prod_{i=1}^T (1 + r_i^B). \quad (78)$$

This is used to calculate the optimal C for a mean-variance criterion for given κ . We take the perspective of the ‘‘average’’ member and we therefore perform the optimization for a fund in

⁹The partial result that the funding ratio given no bonus does not converge to one can be obtained without too much effort. Loosely speaking, it follows since the bonus waiting time has exponential moments, cf. footnote 4, and since a bonus can only be attributed when the funding ratio is ‘‘close’’ to κ the year prior. Consequently, the probability that the funding ratio is above a certain level can be bounded away from zero at least along a sub-sequence. Essentially the same conditions (exponential moments of time to absorption and increasing absorption times from states far away) are used in Ferrari et al. (1995) to show the existence of a quasi-stationary distribution for a continuous-time Markov chain on a discrete state space. Also note Martinez et al. (1998) which studies quasi-stationarity of a Brownian motion conditioned to stay positive. In our setup this can be seen as the limiting case where bonuses are attributed continuously ($\Delta \approx 0$).

stationarity. The analysis proceeds in two steps. First, we calculate the mean and variance of O_T for a fund starting at the threshold. Second, based on these results we apply an exact simulation algorithm to find the stationary mean and variance. The algorithm is considerably simpler than existing algorithms and might be of independent interest.

Fund at bonus threshold

The first and second order moment of O_T (and thereby the variance) can be calculated by a so-called last-exit decomposition, see e.g. Meyn and Tweedie (2009) p. 178. Let \mathcal{U} denote the last time a bonus was given before and including time T . Note that \mathcal{U} is not a stopping time, and that the decomposition is not a consequence of the Markov property. We have

$$\begin{aligned} E_\kappa(O_T) &= E_\kappa(O_T; \mathcal{T}_1 > T) + \sum_{j=1}^T E_\kappa(O_T; \mathcal{U} = j) \\ &= \frac{e^{rT}}{\kappa} e_\kappa(T) + \frac{e^{rT}}{\kappa} \sum_{j=1}^T \left[\sum_{\sigma \in \mathcal{D}_j} \tilde{c}_\sigma \prod_{i=1}^j (\tau_i E_\kappa(1 + r_i^B | \mathcal{T}_1 = i))^{\sigma_i} \right] e_\kappa(T - j), \end{aligned} \quad (79)$$

where for $\sigma \in \mathcal{D}_j$ and $n = 0, \dots, T$,

$$\tilde{c}_\sigma = \frac{(\sum_{i=1}^j \sigma_i)!}{\prod_{i=1}^j \sigma_i!}, \quad e_\kappa(n) = P_\kappa(\mathcal{T}_1 > n) E_\kappa(F_n | \mathcal{T}_1 > n). \quad (80)$$

The expression for $E_\kappa(O_T; \mathcal{U} = j)$ follows by considering the different ways bonuses can be attributed such that the last bonus falls at time j . The different patterns of time between bonuses are given by the permutations in \mathcal{D}_j . For each pattern the probability of it occurring and the associated expected bonus can be calculated by the Markov property, and this has to be multiplied by the number of ways the ‘‘bonus waiting periods’’ can be arranged, given by \tilde{c}_σ . Finally, we multiply by the expected funding ratio given that no bonuses are given for the remaining period, given by $e_\kappa(T - j)$. The quantities appearing in expressions (79) and (80) can be calculated by Propositions 4.2–4.4.

For the second order moment we similarly find

$$\begin{aligned} E_\kappa(O_T^2) &= E_\kappa(O_T^2; \mathcal{T}_1 > T) + \sum_{j=1}^T E_\kappa(O_T^2; \mathcal{U} = j) \\ &= \frac{e^{2rT}}{\kappa^2} s_\kappa(T) + \frac{e^{2rT}}{\kappa^2} \sum_{j=1}^T \left[\sum_{\sigma \in \mathcal{D}_j} \tilde{c}_\sigma \prod_{i=1}^j \left(\tau_i E_\kappa \left((1 + r_i^B)^2 | \mathcal{T}_1 = i \right) \right)^{\sigma_i} \right] s_\kappa(T - j), \end{aligned} \quad (81)$$

where \tilde{c}_σ is given by (80) and for $n = 0, \dots, T$,

$$s_\kappa(n) = P_\kappa(\mathcal{T}_1 > n) E_\kappa(F_n^2 | \mathcal{T}_1 > n). \quad (82)$$

In order to calculate $s_\kappa(k)$ note that on the event $(\mathcal{T}_1 > n)$

$$F_n^2 = ((\kappa - 1)e^{-S_n} + 1)^2 = (\kappa - 1)^2 e^{-2S_n} + 2(\kappa - 1)e^{-S_n} + 1. \quad (83)$$

Hence, we need to calculate conditional expectations of e^{-S_n} and e^{-2S_n} . The first of these is given by (67) of Proposition 4.4. The latter can be calculated by the same formula upon replacing the term $E(e^{-S_n}; S_n > 0)$ with $E(e^{-2S_n}; S_n > 0)$. To evaluate (81) we also need to calculate the second order moment of the bonus percentage. Similarly to F_n^2 the term $(1 + r_i^B)^2$ can be expanded and expressed in terms of e^{-S_i} and e^{-2S_i} . The appropriate conditional expectations of the latter quantities can be calculated by the formula (58) of Proposition 4.3. We omit the details.

Fund in stationarity

We are interested in calculating the stationary mean and variance of O_T . From Propositions 4.2–4.4 we know the time and size of the first bonus, moments of the initial funding ratio and moments of the terminal funding ratio conditioned on no bonuses. Unfortunately, we need the joint distribution of these quantities and this is not available in an analytically tractable form. Instead we will apply a simulation algorithm based on samples from the joint stationary distribution of $(F_0, \mathcal{T}_1, r_{\mathcal{T}_1}^B, F_{\mathcal{T}_1 \wedge T})$. The idea is to split the period in two, the time up to the first bonus (if it occurs before time T) and the time after the first bonus. Moments of O_T can be obtained by combining samples for the period up to the first bonus with analytic results for the period after the first bonus.

Samples from the joint distribution of $(F_0, \mathcal{T}_1, r_{\mathcal{T}_1}^B, F_{\mathcal{T}_1 \wedge T})$ can be obtained in several ways. Perhaps the most obvious is to simulate F_0 from the stationarity funding ratio distribution. Given F_0 we can then simulate the evolution of the fund until the first bonus and record the time and size of the bonus. Ensor and Glynn (2000) give an algorithm which can be used to obtain exact samples of F_0 , and Preisel et al. (2010) give an alternative algorithm by which F_0 can be sampled to any desired level of accuracy. Both algorithms rely on the fact that the invariant distribution of a one-sided random walk (the Y -chain of Section 4) equals the distribution of the maximum of the associated unrestricted random walk (the S -chain of Section 4).

We employ a different idea based on the argument presented in Section 4.2. In stationarity the probability that at time 0 we are in a period between two bonuses of length k is proportional to $k\tau_k$. Further, given we are in a period of length k at time 0 it is equally likely that we are in any of the k positions. This observation gives rise to a very simple algorithm for simulating in stationarity: For a fund started *at the threshold* simulate the path up to the first bonus, say, at time k . This happens with probability τ_k . Now, use this path to generate k samples by shifting it n places to the left for $n = 0, \dots, k-1$. Repeat the algorithm to obtain more samples.

The algorithm generates partly dependent samples from the stationary distribution. However, when used to estimate expectations with respect to the stationary distribution the dependence does not pose a problem. For evaluating moments of O_T in stationarity we propose the following method. The method combines the (exact) samples with the analytic results obtained previously to obtain a consistent estimate of $E_\pi(O_T)$. Estimates of second, and higher, order moments of O_T are obtained by suitable modifications of the G_T -functional.

Proposition 5.1. *Let N be given. Starting at the bonus threshold simulate N paths until the first bonus. Denote the funding ratio paths by $(F_0^{(i)}, \dots, F_{\mathcal{T}_1^{(i)}-1}^{(i)})$ and the first bonus by $r^{(i)}$ for $i = 1, \dots, N$. Let $M = \sum_{i=1}^N \mathcal{T}_1^{(i)}$. A consistent estimate of $E_\pi(O_T)$ can be obtained by*

$$\hat{E}_\pi(O_T) = \frac{1}{M} \sum_{i=1}^N \sum_{k=0}^{\mathcal{T}_1^{(i)}-1} G_T \left(F_k^{(i)}, \mathcal{T}_1^{(i)} - k, r^{(i)}, F_{\mathcal{T}_1^{(i)} \wedge (T+k)}^{(i)} \right), \quad (84)$$

where

$$G_T(F_0, \mathcal{T}_1, r^B, F_{\mathcal{T}_1 \wedge T}) = \frac{F_{\mathcal{T}_1 \wedge T}}{F_0} \times \begin{cases} e^{r\mathcal{T}_1} (1 + r^B) E_\kappa(O_{T-\mathcal{T}_1}) & \text{for } \mathcal{T}_1 \leq T, \\ e^{rT} & \text{for } \mathcal{T}_1 > T, \end{cases} \quad (85)$$

and $O_0 = 0$ by convention.

Note that the two cases in G_T correspond to whether or not the first bonus (in the shifted path) occurs before or after time T . Also note that a bonus occurring before or at time T for the k -shifted path is equivalent to $\mathcal{T}_1^{(i)} \geq T + k$, and in this case the last argument of G_T equals κ .

Although Proposition 5.1 is presented as a method for estimating a specific quantity the same method can be used to estimate any stationary expectation. The estimator can also be made unbiased by replacing M by its expectation, $NE_\kappa(\mathcal{T}_1)$. As a simple example, stationary

probabilities for F_0 can be estimated unbiasedly by

$$\hat{P}_\pi(F_0 \in A) = \frac{1}{NE_\kappa(\mathcal{T}_1)} \sum_{i=1}^N \sum_{k=0}^{\mathcal{T}_1^{(i)}-1} 1_A(F_k^{(i)}) \quad (86)$$

for any event A .¹⁰

Mean-variance analysis

We are now in a position to do a mean-variance analysis of the payout O_T with respect to the strategy parameters κ and C . We assume a horizon of $T = 40$ years. This corresponds approximately to the average time between when contributions are made and benefits are paid out in a pension fund with life-long memberships.

We first consider a fund starting at the bonus threshold. Table 3 states the mean, the minimum (guarantee) and the standard deviation of the payout for different sets of (κ, C) . In all cases the mean equals 6. For the smallest value of κ the payout is guaranteed to be at least 2.656. To reach an expected payout of 6 the bonus potential has to be leveraged substantially, and this in turn leads to a large standard deviation. As κ increases the guarantee decreases and an expected payout of 6 can be achieved by investing the bonus potential less aggressively. This reduces the standard deviation at the price of a larger downside risk (lower guarantee). In the limit as κ tends to infinity there is no guarantee and an expected payout of 6 can be achieved with a standard deviation of 2.17 by investing 37 percent of total assets in stocks.¹¹

κ	1.25	1.5	2	3	5	10
C	2.705	1.259	0.782	0.570	0.468	0.413
mean	6	6	6	6	6	6
guarantee	2.656	2.213	1.660	1.107	0.664	0.332
std. dev.	3.662	2.603	2.356	2.256	2.214	2.191

Table 3: Mean, guarantee and standard deviation of O_{40} for a pension fund starting at the bonus threshold. All sets of strategies (κ, C) imply a mean payout of 6.

Consider next a pension fund which has fixed the bonus threshold at κ . This implies that at least $1/\kappa$ of contributions is guaranteed the risk free rate. The value of the threshold might be stipulated by regulation to ensure a certain minimum pension, or it might be decided by the board of the pension fund based on social economic considerations. In either case, the fund needs to determine an investment strategy C . One (common) way to balance the desire for a high payout against unwanted variability is by use of a mean-variance optimization criterion. Being a collective pension fund we want to optimize the fund for the benefit of the average member, i.e. in stationarity. Hence, we consider the following stationary mean-variance problem for fixed κ

$$\sup_C \{E_\pi(O_T) - \gamma \text{Var}_\pi(O_T)\}. \quad (87)$$

In Figure 5 the optimization problem with $\gamma = 0.07468$ and $T = 40$ is illustrated for $\kappa = 1.5$ and $\kappa = 3$. The value of γ is chosen such that 60 percent in stocks is optimal for a mean-variance investor with no guarantee and a constant proportion of total assets in stocks. We see from Figure 5 that neither the mean nor the standard deviation is monotone in C . The expected payout is decreasing for C sufficiently large because very aggressive strategies lead to low funding ratios in stationarity. The fund with low guarantees ($\kappa = 3$) has an optimal C of about 80 percent,

¹⁰Remark: Theorem 10.4.9 of Meyn and Tweedie (2009) together with Kac's theorem yields the representation result $P_\pi(F_0 \in A) = E_\kappa(\sum_{k=0}^{\mathcal{T}_1-1} 1_A(F_k))/E_\kappa(\mathcal{T}_1)$. This result can also be obtained from (86) by taking expectation on both sides. Conversely, it follows from the representation that the right-hand side of (86) is an unbiased estimator of $P_\pi(F_0 \in A)$ as claimed.

¹¹For the optimal mean-variance investment strategy see Korn (1997) and Zhou and Li (2000).

while the fund with high guarantees ($\kappa = 1.5$) has an optimal C of about 150 percent. The equity exposure, at the threshold, as a fraction of total assets is about 50 percent for both cases.

The mean-variance criterion is normally applied in situations where more risk (higher C) leads to higher expected return and higher variability. It might be argued that the mean-variance criterion is considered out of necessity to avoid degenerate solutions in these situations. In stationarity, however, the expected payout as a function of C is unimodal. We might therefore alternatively define the optimal investment strategy C^* as the one maximizing the expected payout in stationarity. The optimal investment strategy C^* thus defined is illustrated in Table 4 and Figure 5 upper left for different values of κ . We see that C^* is considerably higher than the one obtained from the mean-variance criterion (for given κ). However, the associated standard deviation is also considerably higher implying that the payout will typically deviate substantially from its expectation.

κ	1	1.25	1.5	2	3	5	10
C^*	N/A	2.143	2.313	2.473	2.700	2.850	2.951
maximal mean	3.320	4.923	6.886	11.73	23.66	48.50	93.61
std. dev.	0	2.213	6.649	26.13	151.5	826.9	3540

Table 4: The optimal investment strategy, C^* , maximizing the expectation of O_{40} in stationarity for different values of κ . The corresponding mean and standard deviation of O_{40} are also shown.

5.3 Concluding remarks

In this section we have studied a stylized collective pension fund and presented an essentially complete analysis of its stationary properties. We have argued that optimizing the fund’s stationary dynamics is a way to fulfill the otherwise vague objective of “in the best interest of members”. Of course, some (the board) must still decide on the collective risk appetite. The analysis is intricate and the model under study is kept simple of mathematical necessity. However, we believe that the lessons learned have far-reaching real world consequences.

The simplest and most important lesson is that there is a limit to how much risk the fund can take, and this limit can be surprisingly low (Proposition 4.1). When this limit is exceeded the fund will over time lose its bonus potential. Although the specific limit depends on the model, this is a general result that applies to all investors with a nonrenewable risk budget. It shows that care must be taken when introducing volatility on the balance, and it also provides a rationale for funds to buy tail protection to curb losses.

The second lesson of general applicability is that there is a genuine trade-off between short-term gains and long-term funding. In the long run too little investment risk leads to frequent, small bonuses, while too much investment risk leads to infrequent, large bonuses. Taken to the extreme too little and too much risk both lead to vanishing long-term bonuses and a balance must therefore be struck, as illustrated in the lower two plots of Figure 4. This trade-off exists for all long-term investors trying to add value to a fixed liability from a limited amount of “free money”. The key insight is that high expected returns come with the risk of big losses which in turn impair future risk taking.

The third lesson is that bonus “deserts” cannot be prevented. Even if the fund is risk averse there will be substantial variation in the time between bonuses, and this variability increases with the investment risk, see Table 2 or Figure 3 lower right plot. The sheer magnitude of the variability is perhaps the most surprising result of the paper. Bearing in mind that the analysis is in some sense an idealized “best case” we would expect that when the underlying assumptions are violated the time between bonuses will be even wider.

The final lesson to be “learned” is the analysis methodology in itself. Stationarity is a mathematical abstraction which we use to represent the long-term going-concern of a pension fund.

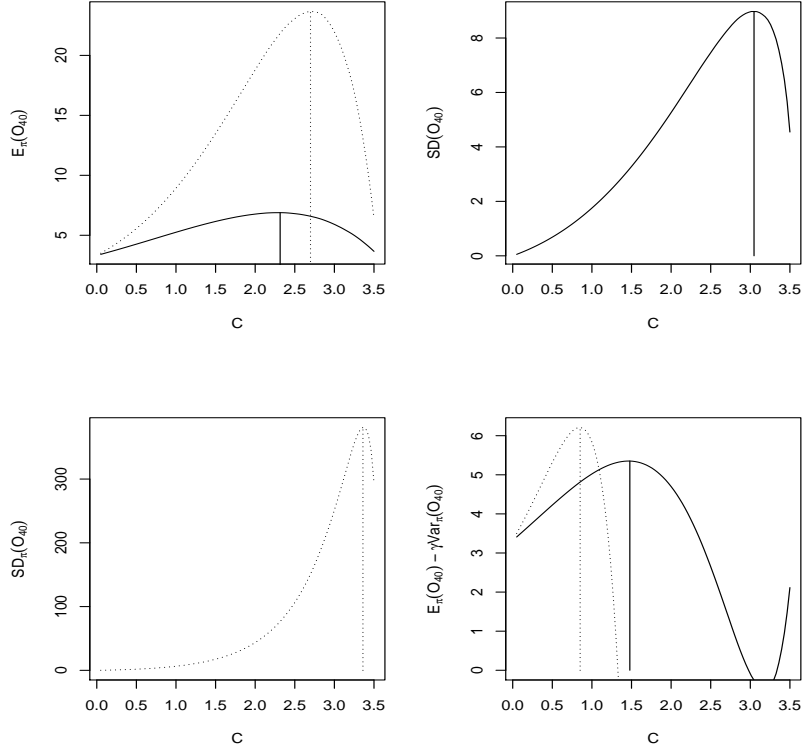


Figure 5: Stationary mean and standard deviation of O_{40} as a function of the investment strategy C . Solid curves correspond to $\kappa = 1.5$ and dotted curves to $\kappa = 3$. Upper left: Expected pension payout. Upper right and lower left: Standard deviation of the pension payout (note the different scales). Lower right: Illustration of the mean-variance optimization problem (87).

Although stationarity in the strict probabilistic sense will never be achieved in practice, we believe that collective pension funds should be designed with the aim of perpetual operation, i.e. with stationary dynamics. The largest Danish pension fund, ATP, has successfully applied these ideas in practice for more than a decade when designing policies for risk budgeting, hedging, investment and indexation, as well as the pension product itself.

The work can be extended in a number of directions to make the model more realistic. A stochastic interest rate model would introduce interest rate sensitivity to the balance and influence the dynamics of the funding ratio process. In practice, interest rate fluctuations has probably been the single most important factor for the funding ratio of many pension funds in recent years. It would also be of considerable interest to introduce a demographic model and study the impact of the gradual aging of the population that follows from increased life expectancy.

In practice pension funds, and investors in general, do not operate in perfect markets. Rather, they face a wide range of frictions in the form of e.g. regulatory requirements, leverage constraints, liquidity risk, different rates for loans and deposits, currency risk, operational risk etc. Each of these effects are of interest in their own right, and it could be relevant to include these in the analysis. However, non-hedgeable risks preclude the existence of a stationary solution and call for an alternative object of study. To stay in the spirit of the current work, one possibility would be to demand a low probability of insolvency on a given horizon and to study the dynamics of the fund conditioned on “survival”.

Another avenue of research is the capital market model and the investment strategy. In

particular, it would be of interest to study capital markets with heavy-tailed returns and option based investment strategies. The random walk results of Section 2 are general results which apply to any funding ratio process of the form

$$F_n = \min \{ (F_{n-1} - 1)e^{-X_n} + 1, \kappa \} \text{ for } n \in \mathbb{N}_0, \quad (88)$$

where the X_n 's are i.i.d. random variables. Note that X_n represents the joint effect of the capital market return and the investment strategy. In this paper X_n follows a normal distribution, but in principle any distribution can be assumed. However, to apply the results we need the distribution of $S_n = X_1 + \dots + X_n$ so in practice only distributions with known convolution properties can be handled analytically, e.g. Gamma or compound Poisson distributions. Otherwise we have to resort to numerical methods.

With fat-tailed returns or jumps in the market large losses can occur over short periods of time, or even instantly. With no limits on the size of losses and if the fund does not have access to tail protection, the fund can stay funded with certainty only if it does not lever its bonus potential. On the other hand, if tail protection is available the fund can still use leverage, but the exposure in excess of the bonus potential has to be protected. In general, the risk of big losses causes the fund to behave more cautiously and consequently it will face lower expected returns in the long run.

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A Proofs

A.1 Entrance times and partitions

Proof of Theorem 2.3. Rearranging (5) we get for $0 \leq s < 1$

$$1 - \exp\left(\sum_{n=1}^{\infty} (-1) \frac{p_n}{n} s^n\right) = \sum_{n=1}^{\infty} \tau_n s^n. \quad (89)$$

From Theorem 2.2 we get that

$$\begin{aligned} \exp\left(\sum_{n=1}^{\infty} (-1) \frac{p_n}{n} s^n\right) &= 1 + \sum_{m=1}^{\infty} \sum_{\sigma \in \mathcal{D}_m} \frac{s^m}{\sigma_1! \sigma_2! \cdots \sigma_m!} \prod_{n=1}^m \left((-1) \frac{p_n}{n}\right)^{\sigma_n} \\ &= 1 + \sum_{m=1}^{\infty} \sum_{\sigma \in \mathcal{D}_m} \frac{\text{sgn}(\sigma)}{d_\sigma} s^m \prod_{n=1}^m p_n^{\sigma_n}. \end{aligned} \quad (90)$$

Insert this in (89) to obtain

$$-\sum_{m=1}^{\infty} \sum_{\sigma \in \mathcal{D}_m} \frac{\text{sgn}(\sigma)}{d_\sigma} s^m \prod_{n=1}^m p_n^{\sigma_n} = \sum_{n=1}^{\infty} \tau_n s^n. \quad (91)$$

Finally, by inspection of the left and the right hand side of the above equality we obtain (12). \square

Proof of Theorem 2.4. We need to prove two identities

$$\sum_{\sigma \in \mathcal{D}_n} \frac{k\sigma_k}{d_\sigma} = 1, \quad (92)$$

and

$$\sum_{\sigma \in \mathcal{D}_n} \frac{1}{d_\sigma} = 1. \quad (93)$$

We first note that (93) follows from (92) by summing over k . Using that $\sum_{k=1}^n k\sigma_k = n$ for $\sigma \in \mathcal{D}_n$ we get

$$n = \sum_{k=1}^n \sum_{\sigma \in \mathcal{D}_n} \frac{k\sigma_k}{d_\sigma} = \sum_{\sigma \in \mathcal{D}_n} \frac{\sum_{k=1}^n k\sigma_k}{d_\sigma} = \sum_{\sigma \in \mathcal{D}_n} \frac{n}{d_\sigma}, \quad (94)$$

and dividing throughout by n yields (93).

We next prove (92) by induction in n . For $n = 1$ the relation is trivially satisfied. Assume that (92) holds for $n - 1$ and all $1 \leq k \leq n - 1$; then (93) also holds for $n - 1$ as just shown. To prove that (92) holds for n and $1 \leq k \leq n$ there are three cases to consider:

For $k = n$ the only term in the sum different from zero occurs for $\sigma = (0, \dots, 0, 1)$, and hence

$$\sum_{\sigma \in \mathcal{D}_n} \frac{n\sigma_n}{\prod_{i=1}^n \sigma_i! i^{\sigma_i}} = \frac{n}{1!n} = 1. \quad (95)$$

For $1 < k < n$ we use that there is a one-to-one mapping between permutations in \mathcal{D}_n with $\sigma_k > 0$ and permutations in \mathcal{D}_{n-1} with $\pi_{k-1} > 0$ defined by $\pi = (\sigma_1, \dots, \sigma_{k-2}, \sigma_{k-1} + 1, \sigma_k -$

$1, \sigma_{k+1}, \dots, \sigma_{k-1}$). Noting that $\sigma_k > 0$ with $k < n$ implies $\sigma_n = 0$ we have

$$\begin{aligned}
& \sum_{\sigma \in \mathcal{D}_n} \frac{k\sigma_k}{\prod_{i=1}^n \sigma_i! i^{\sigma_i}} \\
&= \sum_{\sigma \in \mathcal{D}_n: \sigma_k > 0} \frac{k\sigma_k}{\prod_{i=1}^{n-1} \sigma_i! i^{\sigma_i}} \\
&= \sum_{\pi \in \mathcal{D}_{n-1}: \pi_{k-1} > 0} \frac{k(\pi_k + 1)}{\prod_{i=1}^{n-1} \pi_i! i^{\pi_i}} \frac{\pi_{k-1}!(k-1)^{\pi_{k-1}} \pi_k! k^{\pi_k}}{(\pi_{k-1}-1)!(k-1)^{\pi_{k-1}-1}(\pi_k+1)! k^{\pi_k+1}} \\
&= \sum_{\pi \in \mathcal{D}_{n-1}: \pi_{k-1} > 0} \frac{(k-1)\pi_{k-1}}{\prod_{i=1}^{n-1} \pi_i! i^{\pi_i}} \\
&= 1,
\end{aligned} \tag{96}$$

where the last equality follows by the induction hypothesis.

For $k = 1$ the mapping $\pi = (\sigma_1 - 1, \sigma_2, \dots, \sigma_{n-1})$ is one-to-one between permutations in \mathcal{D}_n with $\sigma_1 > 0$ and *all* permutations in \mathcal{D}_{n-1} . We then have

$$\sum_{\sigma \in \mathcal{D}_n} \frac{\sigma_1}{\prod_{i=1}^n \sigma_i! i^{\sigma_i}} = \sum_{\pi \in \mathcal{D}_{n-1}} \frac{\pi_1 + 1}{\prod_{i=1}^{n-1} \pi_i! i^{\pi_i}} \frac{\pi_1!}{(\pi_1 + 1)!} = \sum_{\pi \in \mathcal{D}_{n-1}} \frac{1}{\prod_{i=1}^{n-1} \pi_i! i^{\pi_i}} = 1, \tag{97}$$

where the last equality follows from (93) which holds for $n - 1$ by the induction hypothesis. \square

A.2 Entrance time moments

Proof of Theorem 2.5. Let $G_1 = H'$, where H is defined in (6), and define recursively

$$G_n = H'G_{n-1} + G'_{n-1} \quad (n \geq 2). \tag{98}$$

We first establish the relation

$$\tau^{(n)} = -e^H G_n \quad (n \geq 1). \tag{99}$$

Clearly, the relation holds for $n = 1$. Assuming (99) holds for $n - 1$ we have by (98)

$$\begin{aligned}
\tau^{(n)} &= (-e^H G_{n-1})' \\
&= -e^H (H'G_{n-1} + G'_{n-1}) \\
&= -e^H G_n,
\end{aligned} \tag{100}$$

and hence (99) holds for all n by induction.

Having established (99) we now need to prove

$$G_n = \sum_{\sigma \in \mathcal{D}_n} c_\sigma H_\sigma \quad (n \geq 1). \tag{101}$$

As before we will prove this by induction. For $n = 1$ the equation reads $G_1 = H'$ which is true by definition. Assume (101) holds for $n - 1$. By (98) and the induction hypothesis we have

$$G_n = \sum_{\sigma \in \mathcal{D}_{n-1}} c_\sigma H' H_\sigma + \sum_{\sigma \in \mathcal{D}_{n-1}} c_\sigma (H_\sigma)', \tag{102}$$

where for $\sigma = (\sigma_1, \dots, \sigma_{n-1}) \in \mathcal{D}_{n-1}$

$$\begin{aligned}
(H_\sigma)' &= \sigma_1 H_1^{\sigma_1-1} H_2^{\sigma_2} \prod_{i=2}^{n-1} H_i^{\sigma_i} + H_1^{\sigma_1} \left(\prod_{i=2}^{n-1} H_i^{\sigma_i} \right)' \\
&= \dots \\
&= \sum_{k=1}^{n-1} \sigma_k H_k^{\sigma_k-1} H_{k+1} \prod_{i \neq k} H_i^{\sigma_i}.
\end{aligned} \tag{103}$$

Thus

$$G_n = \sum_{\sigma \in \mathcal{D}_{n-1}} c_\sigma H' H_\sigma + \sum_{\sigma \in \mathcal{D}_{n-1}} \sum_{k=1}^{n-1} c_\sigma \sigma_k H_k^{\sigma_k-1} H_{k+1} \prod_{i \neq k} H_i^{\sigma_i}. \quad (104)$$

Let $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*) \in \mathcal{D}_n$ be fixed, but arbitrary. We will identify the terms in (104) which contain the factor H^{σ^*} and show that the coefficients add up to c_{σ^*} . There are three cases (note that in the first two cases the condition implies that $\sigma_n^* = 0$):

1. $\sigma_1^* > 0$: Then $H' H_\sigma = H^{\sigma^*}$ with $\sigma = (\sigma_1^* - 1, \sigma_2^*, \dots, \sigma_{n-1}^*) \in \mathcal{D}_{n-1}$. The first sum in (104) contains a term with this factor and coefficient c_σ .
2. $\sigma_j^* > 0$ for some $2 \leq j \leq n-1$: Then $H_k^{\sigma_k-1} H_{k+1} \prod_{i \neq k} H_i^{\sigma_i} = H^{\sigma^*}$ with $\sigma = (\sigma_1^*, \dots, \sigma_{j-2}^*, \sigma_{j-1}^* + 1, \sigma_j^* - 1, \sigma_{j+1}^*, \dots, \sigma_{n-1}^*) \in \mathcal{D}_{n-1}$ and $k = j-1$. The second sum in (104) contains a term with this factor and coefficient $c_\sigma(\sigma_{j-1}^* + 1)$.
3. $\sigma_n^* > 0$: In this case $\sigma^* = (0, \dots, 0, 1)$ and hence $H^{\sigma^*} = H_n = H_k^{\sigma_k-1} H_{k+1} \prod_{i \neq k} H_i^{\sigma_i}$ for $\sigma = (0, \dots, 0, 1) \in \mathcal{D}_{n-1}$ and $k = n-1$. This term is included in the second sum of (104) with coefficient c_σ .

Hence, for each positive component of σ^* there exists a term in (104) containing the factor H^{σ^*} . Conversely, every term in (104) corresponds to a $\sigma^* \in \mathcal{D}_n$ and is of the form covered in one of the three cases above. Hence, we have

$$G_n = \sum_{\sigma^* \in \mathcal{D}_n} H^{\sigma^*} \sum_{1 \leq j \leq n: \sigma_j^* > 0} (\sigma_{j-1}^* + 1) c_{\sigma^{(j)}} \quad (105)$$

where $\sigma^{(j)} = (\sigma_1^*, \dots, \sigma_{j-2}^*, \sigma_{j-1}^* + 1, \sigma_j^* - 1, \sigma_{j+1}^*, \dots, \sigma_{n-1}^*) \in \mathcal{D}_{n-1}$ and we, for notational convenience, define $\sigma_0^* \equiv 0$. Using (10) we can write the inner sum in (105) as

$$\begin{aligned} & \sum_{1 \leq j \leq n: \sigma_j^* > 0} (\sigma_{j-1}^* + 1) c_{\sigma^{(j)}} \\ &= \sum_{1 \leq j \leq n: \sigma_j^* > 0} (\sigma_{j-1}^* + 1) \frac{(n-1)!}{\prod_{i=1}^{n-1} \sigma_i^{(j)}! (i!)^{\sigma_i^{(j)}}} \\ &= \frac{1}{\prod_{i=1}^n \sigma_i^*! (i!)^{\sigma_i^*}} \sum_{1 \leq j \leq n: \sigma_j^* > 0} (\sigma_{j-1}^* + 1) (n-1)! \frac{\prod_{i=1}^n \sigma_i^*! (i!)^{\sigma_i^*}}{\prod_{i=1}^{n-1} \sigma_i^{(j)}! (i!)^{\sigma_i^{(j)}}} \\ &= \frac{1}{\prod_{i=1}^n \sigma_i^*! (i!)^{\sigma_i^*}} \sum_{1 \leq j \leq n: \sigma_j^* > 0} (\sigma_{j-1}^* + 1) (n-1)! \frac{\sigma_j^* j!}{(\sigma_{j-1}^* + 1)(j-1)!} \\ &= \frac{1}{\prod_{i=1}^n \sigma_i^*! (i!)^{\sigma_i^*}} (n-1)! \sum_{1 \leq j \leq n: \sigma_j^* > 0} \sigma_j^* j \\ &= \frac{n!}{\prod_{i=1}^n \sigma_i^*! (i!)^{\sigma_i^*}} \\ &= c_{\sigma^*}, \end{aligned} \quad (106)$$

where the penultimate equality uses that $\sum_{1 \leq j \leq n: \sigma_j^* > 0} \sigma_j^* j = n$ since $\sigma^* \in \mathcal{D}_n$. This shows that (101) holds and we are finished. \square

Before proving Theorem 2.6 we need to state the following two lemmas:

Lemma A.1. For $n \geq 1$, $(a_1, \dots, a_n) \in \mathbb{C}^n$ and $b \in \mathbb{C}$

$$\sum_{\sigma \in \mathcal{D}_n} \prod_{i=1}^n \left(a_i - \frac{b^i}{i} \right)^{\sigma_i} \frac{1}{\sigma_i!} = \sum_{\sigma \in \mathcal{D}_n} \prod_{i=1}^n \frac{a_i^{\sigma_i}}{\sigma_i!} - b \sum_{\sigma \in \mathcal{D}_{n-1}} \prod_{i=1}^{n-1} \frac{a_i^{\sigma_i}}{\sigma_i!}, \quad (107)$$

where the last sum is 1 by definition for $n = 1$.

Proof. We first note that the left-hand side of (107) can be written

$$\sum_{\sigma \in \mathcal{D}_n} \prod_{i=1}^n \frac{a_i^{\sigma_i}}{\sigma_i!} - b \sum_{\sigma \in \mathcal{D}_n} \sigma_1 \frac{a_1^{\sigma_1-1}}{\sigma_1!} \prod_{i=2}^n \frac{a_i^{\sigma_i}}{\sigma_i!} + \dots, \quad (108)$$

where \dots denotes higher order terms of b . Since there is only a contribution to the first order term if $\sigma_1 > 0$, and since $\sigma \in \mathcal{D}_n$ with $\sigma_1 > 0$ if and only if $(\sigma_1 - 1, \sigma_2, \dots, \sigma_{n-1}) \in \mathcal{D}_{n-1}$, we have that the first order term is given by

$$b \sum_{\sigma \in \mathcal{D}_n: \sigma_1 > 0} \sigma_1 \frac{a_1^{\sigma_1-1}}{\sigma_1!} \prod_{i=2}^n \frac{a_i^{\sigma_i}}{\sigma_i!} = b \sum_{\sigma \in \mathcal{D}_{n-1}} \prod_{i=1}^{n-1} \frac{a_i^{\sigma_i}}{\sigma_i!}. \quad (109)$$

Hence we are finished if we show that all higher order terms in (108) vanish.

For $m \geq 2$, the m 'th order terms are of the form

$$\prod_{i=1}^n a_i^{\eta_i} \left(\frac{-b^i}{i} \right)^{\nu_i} \frac{1}{(\eta_i + \nu_i)!} \binom{\eta_i + \nu_i}{\nu_i} = \frac{\text{sgn}(\nu)}{d_\nu} b^m \prod_{i=1}^n \frac{a_i^{\eta_i}}{\eta_i!}, \quad (110)$$

where $\nu \in \mathcal{D}_m$ and $\eta \in \mathcal{D}_{n-m}$, with the convention that $\nu_j = 0$ for $j > m$ and, similarly, $\eta_j = 0$ for $j > n - m$ (for $m = n$, $\eta_j = 0$ for all j). The higher order terms in (108) can then be written

$$\sum_{m=2}^n b^m \sum_{\eta \in \mathcal{D}_{n-m}} \prod_{i=1}^n \frac{a_i^{\eta_i}}{\eta_i!} \sum_{\nu \in \mathcal{D}_m} \frac{\text{sgn}(\nu)}{d_\nu} = 0, \quad (111)$$

since the inner sum is zero for $m \geq 2$ by equation (13). \square

We also need the following combination of Kronecker's lemma, see e.g. Theorem 2.5.5 of Durrett (2010), and Frobenius' theorem, see e.g. Chapter 7 of Duren (2012).

Lemma A.2. *If $\sum_{k=1}^{\infty} a_k$ converges to a finite limit then*

$$f(s) = (1-s) \sum_{k=1}^{\infty} s^k k a_k \quad (112)$$

converges for $|s| < 1$ and $f(s) \rightarrow 0$ as $s \rightarrow 1-$.

Proof. Let $b_k = k$ and $x_k = k a_k$. Then $b_k \uparrow \infty$ and, by assumption, $\sum_{k=1}^{\infty} x_k / b_k = \sum_{k=1}^{\infty} a_k$ converges. By Kronecker's lemma this implies that

$$\frac{1}{b_n} \sum_{k=1}^n x_k = \frac{1}{n} \sum_{k=1}^n k a_k \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (113)$$

Next, let $a_0 = 0$ and $c_k = k a_k - (k-1) a_{k-1}$ for $k \geq 1$. Summation by parts yields

$$f(s) = (1-s) \sum_{k=1}^{\infty} s^k k a_k = \sum_{k=1}^{\infty} s^k (k a_k - (k-1) a_{k-1}) = \sum_{k=1}^{\infty} s^k c_k, \quad (114)$$

where the series $\sum_{k=1}^{\infty} c_k$ is Cesàro summable to zero by (113), i.e. the average of the partial sums, $\sum_{m=1}^k c_m = k a_k$, converges to zero. By Frobenius' theorem we conclude that $f(s)$ converges for $|s| < 1$ and $f(s) \rightarrow 0$ as $s \rightarrow 1-$. \square

With those two lemmas available we are now ready to prove Theorem 2.6.

Proof of Theorem 2.6. By Theorem 2.5 we have for $|s| < 1$

$$\tau^{(n)}(s) = -e^{H(s)} \sum_{\sigma \in \mathcal{D}_n} c_\sigma H_\sigma(s), \quad (115)$$

where $H(s) = -\sum_{k=1}^{\infty} \frac{s^k}{k} P(S_k \leq 0)$ and

$$H_\sigma(s) = \prod_{i=1}^n \left(-\sum_{k=i}^{\infty} \frac{(k)_i}{k} s^{k-i} P(S_k \leq 0) \right)^{\sigma_i}. \quad (116)$$

Using that $\log(1-s) = -\sum_{k=1}^{\infty} s^k/k$, and hence $(i-1)!/(1-s)^i = \sum_{k=i}^{\infty} s^{k-i} (k)_i/k$, we get

$$\begin{aligned} e^{H(s)} &= \exp \left(\sum_{k=1}^{\infty} \frac{s^k}{k} (1 - P(S_k \leq 0)) - \sum_{k=1}^{\infty} \frac{s^k}{k} \right) \\ &= \exp \left(\sum_{k=1}^{\infty} \frac{s^k}{k} P(S_k > 0) \right) (1-s), \end{aligned} \quad (117)$$

and

$$\begin{aligned} c_\sigma H_\sigma(s) &= n! \prod_{i=1}^n \left(\sum_{k=i}^{\infty} \frac{(k)_i}{k} s^{k-i} (1 - P(S_k \leq 0)) - \sum_{k=i}^{\infty} \frac{(k)_i}{k} s^{k-i} \right)^{\sigma_i} \frac{1}{\sigma_i! (i!)^{\sigma_i}} \\ &= n! \prod_{i=1}^n \left(\sum_{k=i}^{\infty} \frac{(k)_i}{k} s^{k-i} P(S_k > 0) - \frac{(i-1)!}{(1-s)^i} \right)^{\sigma_i} \frac{1}{\sigma_i! (i!)^{\sigma_i}} \\ &= n! \prod_{i=1}^n \left(a_i - \frac{b^i}{i} \right)^{\sigma_i} \frac{1}{\sigma_i!}, \end{aligned} \quad (118)$$

where $a_i = \sum_{k=i}^{\infty} \frac{(k)_i}{k} s^{k-i} P(S_k > 0)/i!$ and $b = 1/(1-s)$. By Lemma A.1 we then have

$$\sum_{\sigma \in \mathcal{D}_n} c_\sigma H_\sigma(s)/n! = \sum_{\sigma \in \mathcal{D}_n} \prod_{i=1}^n \frac{a_i^{\sigma_i}}{\sigma_i!} - b \sum_{\sigma \in \mathcal{D}_{n-1}} \prod_{i=1}^{n-1} \frac{a_i^{\sigma_i}}{\sigma_i!}. \quad (119)$$

Theorem 2.5 in a combination with (117) and (119) yields

$$\begin{aligned} \tau^{(n)}(s) &= \exp \left(\sum_{k=1}^{\infty} \frac{s^k}{k} P(S_k > 0) \right) \sum_{\sigma \in \mathcal{D}_{n-1}} n c_\sigma \prod_{i=1}^{n-1} \left(\sum_{k=i}^{\infty} \frac{(k)_i}{k} s^{k-i} P(S_k > 0) \right)^{\sigma_i} \\ &\quad - \exp \left(\sum_{k=1}^{\infty} \frac{s^k}{k} P(S_k > 0) \right) \sum_{\sigma \in \mathcal{D}_n} (1-s) c_\sigma \prod_{i=1}^n \left(\sum_{k=i}^{\infty} \frac{(k)_i}{k} s^{k-i} P(S_k > 0) \right)^{\sigma_i}. \end{aligned} \quad (120)$$

By assumption $\sum_{k=1}^{\infty} k^{n-2} P(S_k > 0) < \infty$, and by dominated convergence, all series of order at most $n-2$ thereby converge to finite limits as s tends to $1-$. In a combination with Lemma A.2 we also have

$$\lim_{s \rightarrow 1-} (1-s) \sum_{k=n}^{\infty} \frac{(k)_n}{k} s^{k-n} P(S_k > 0) = 0, \quad (121)$$

and we conclude

$$E((\tau_-)_n) = \lim_{s \rightarrow 1-} \tau^{(n)}(s) = \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} P(S_k > 0) \right) n! \sum_{\sigma \in \mathcal{D}_{n-1}} \prod_{i=1}^{n-1} \left(\frac{h_i}{i!} \right)^{\sigma_i} \frac{1}{\sigma_i!} < \infty. \quad (122)$$

□

A.3 Conditional characteristic functions

Proof of Theorem 2.8. Define the (partial) characteristic functions

$$\gamma_n(\zeta) = E(e^{i\zeta S_n}; \tau_- = n), \quad (123)$$

in terms of which χ can be written as

$$\chi(s, \zeta) = \sum_{n=1}^{\infty} s^n \gamma_n(\zeta). \quad (124)$$

Rearranging (26) we get

$$1 - \exp\left(-\sum_{n=1}^{\infty} \frac{s^n}{n} g_n\right) = \sum_{n=1}^{\infty} s^n \gamma_n(\zeta), \quad (125)$$

where we have defined $g_n = E(e^{i\zeta S_n}; S_n \leq 0)$. From Theorem 2.2 we get that

$$\begin{aligned} \exp\left(\sum_{n=1}^{\infty} (-1) \frac{g_n}{n} s^n\right) &= 1 + \sum_{m=1}^{\infty} \sum_{\sigma \in \mathcal{D}_m} \frac{s^m}{\sigma_1! \sigma_2! \cdots \sigma_m!} \prod_{n=1}^m \left((-1) \frac{g_n}{n}\right)^{\sigma_n} \\ &= 1 + \sum_{m=1}^{\infty} \sum_{\sigma \in \mathcal{D}_m} \frac{\text{sgn}(\sigma)}{d_\sigma} s^m \prod_{n=1}^m g_n^{\sigma_n}. \end{aligned} \quad (126)$$

Insert this in (125) to obtain

$$-\sum_{m=1}^{\infty} \sum_{\sigma \in \mathcal{D}_m} \frac{\text{sgn}(\sigma)}{d_\sigma} s^m \prod_{n=1}^m g_n^{\sigma_n} = \sum_{n=1}^{\infty} s^n \gamma_n(\zeta). \quad (127)$$

By inspection of the left and the right hand side of the equality above we obtain

$$\gamma_n(\zeta) = -\sum_{\sigma \in \mathcal{D}_n} \frac{\text{sgn}(\sigma)}{d_\sigma} \prod_{k=1}^n (E(e^{i\zeta S_k}; S_k \leq 0))^{\sigma_k}. \quad (128)$$

Finally, divide by τ_n to obtain (27). \square

Proof of Theorem 2.9. For fixed ζ and n we have the following first-entrance decomposition

$$\begin{aligned} E(e^{i\zeta S_n}) &= \sum_{k=1}^n E(e^{i\zeta S_n}; \tau_- = k) + E(e^{i\zeta S_n}; \tau_- > n) \\ &= \sum_{k=1}^n E(e^{i\zeta S_k}; \tau_- = k) E(e^{i\zeta S_{n-k}}) + E(e^{i\zeta S_n}; \tau_- > n), \end{aligned} \quad (129)$$

where the second equality follows from the Markov property and the random walk structure. For ease of notation we let $e_k = E(e^{i\zeta S_k})$. In this notation, the decomposition above can be written

$$E(e^{i\zeta S_n}; \tau_- > n) = e_n - \sum_{k=1}^n E(e^{i\zeta S_k}; \tau_- = k) e_{n-k}. \quad (130)$$

Multiplying both sides of (130) by e_1 yields the relation

$$\begin{aligned} E(e^{i\zeta S_n}; \tau_- > n) e_1 &= e_{n+1} - \sum_{k=1}^n E(e^{i\zeta S_k}; \tau_- = k) e_{n+1-k} \\ &= E(e^{i\zeta S_{n+1}}; \tau_- > n+1) + E(e^{i\zeta S_{n+1}}; \tau_- = n+1), \end{aligned} \quad (131)$$

where we have used that for all k , $e_k e_1 = e_{k+1}$.

We are now ready to prove (28) by induction in n . Since the events $(\tau_- > 1)$ and $(S_1 > 0)$ are identical equation (28) holds for $n = 1$. Next, assume that (28) holds for n . Let $e_k^+ = E(e^{i\zeta S_k}; S_k > 0)$ and note that $E(e^{i\zeta S_k}; S_k \leq 0) = e_k - e_k^+$. From Theorem 2.8 we have

$$E(e^{i\zeta S_{n+1}}; \tau_- = n+1) = - \sum_{\sigma \in \mathcal{D}_{n+1}} \prod_{k=1}^{n+1} \left(\frac{e_k^+ - e_k}{k} \right)^{\sigma_k} \frac{1}{\sigma_k!}. \quad (132)$$

Using that $e_k = e_1^k$ we get from (131), the induction hypothesis and (132) that

$$\begin{aligned} E(e^{i\zeta S_{n+1}}; \tau_- > n+1) &= e_1 \sum_{\sigma \in \mathcal{D}_n} \prod_{k=1}^n \left(\frac{e_k^+}{k} \right)^{\sigma_k} \frac{1}{\sigma_k!} + \sum_{\sigma \in \mathcal{D}_{n+1}} \prod_{k=1}^{n+1} \left(\frac{e_k^+ - e_1^k}{k} \right)^{\sigma_k} \frac{1}{\sigma_k!} \\ &= \sum_{\sigma \in \mathcal{D}_{n+1}} \prod_{k=1}^{n+1} \left(\frac{e_k^+}{k} \right)^{\sigma_k} \frac{1}{\sigma_k!}, \end{aligned} \quad (133)$$

where the second equality uses Lemma A.1 with $a_k = e_k^+/k$ and $b = e_1$. This shows that (28) holds for $n+1$ and we are finished. \square